Algebra for Elementary Schools

Part III

15

By

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FOR

ELEMENTARY SCHOOLS

PART III.

BY

H. S. HALL, M.A.
JOINT AUTHOR OF HALL AND KNIGHT'S "ELEMENTARY ALGEBRA"
AND HALL AND STEVENS' "EUCLID"

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First Edition 1901.
Reprinted 1902, 1906, 1908, 1913.

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ALGEBRA.

CHAPTER XIX.

SIMULTANEOUS EQUATIONS.

163. In Chap. x. we explained how problems may be solved by means of equations containing *one* unknown quantity; but it frequently happens that the solution of a problem cannot be effected without the use of *two* or *more* unknown quantities. We shall now explain the methods by which the values of more than one unknown quantity may be determined from given equations.

164. Consider the equation $x+y=7$, which contains the two unknown quantities x and y.

Here, since $x=7-y$, it is clear that, for every value we choose to give to y, there will be a corresponding value of x.

Thus, if $y=0$, x must equal 7.
If $y=8$, x „ „ -1.
If $y=2\frac{1}{2}$, x „ „ $4\frac{1}{2}$, and so on.

165. If the values of x and y, which satisfy the equation

$$x+y=7 \quad\text{.................................}(1),$$

also satisfy another equation of the same kind, such as

$$x-5y=13 \quad\text{...............................}(2),$$

we shall find there is only *one solution.*

For from (1) we have $x=7-y$,
and from (2) we have $x=13+5y$;

and since the values of x in the two equations are to be the same, we must have

$$13+5y=7-y;$$

whence $6y=-6$;

that is, $y=-1$.

If we now substitute this value for y in *either* of the two given equations, we obtain $x=8$.

Thus $x=8$, $y=1$ is the only solution possible if the two equations are to be satisfied by the *same* pair of values of x and y.

166. DEFINITION. When two or more equations are satisfied by the same values of the unknown quantities, they are called **Simultaneous Equations.**

167. Since such equations are simultaneously true, it is plain that *any* equation formed by combining them will also be true for those values of the unknowns which satisfy the original equations. In combining the given equations our first object will always be to obtain an equation which involves only *one* of the unknown quantities.

168. The process by which we get rid of an unknown quantity is called **elimination,** and it must be effected in different ways according to the nature of the equations given for solution.

First Method.

Example 1. Solve

$$3x+7y=27 \quad \text{.............................(1)},$$
$$5x+2y=16 \quad \text{.............................(2)}.$$

To eliminate x we multiply (1) by 5 and (2) by 3, so as to make the coefficients of x in both equations equal. This gives

$$15x+35y=135,$$
$$15x+\ 6y=\ 48;$$

subtracting,

$$29y=\ 87;$$
$$\therefore\ y=3.$$

To find x, substitute this value of y in *either* of the given equations.

Thus from (1)

$$3x+21=27;$$
$$\therefore\ x=2.$$

Therefore the complete solution is $x=2$, $y=3$.

Note. When one of the unknowns has been found, it is immaterial which of the equations we use to complete the solution. Thus, in the present example, if we substitute 3 for y in (2), we have

$$5x+6=16;$$
$$\therefore\ x=2, \text{ as before.}$$

Example 2. Solve $7x+2y=47$ ....(1),

$5x-4y=1$ ....(2).

Here it will be more convenient to eliminate y.

Multiplying (1) by 2, $14x+4y=94$,

and from (2), $5x-4y=1$;

adding, $19x=95$;

$$\therefore\ x=5.$$

Substituting this value in (1),

$$35+2y=47;$$

$$\therefore\ y=6,$$

and $x=5$.

Note 1. By comparing the second step in Example 2 with that in Example 1, the pupil will see that he must *add* when the coefficients of one unknown are equal and *unlike* in sign, and *subtract* when the coefficients are equal and *like* in sign.

Note 2. The solution should be verified by substituting in *both* equations the values obtained for the unknown quantities.

EXAMPLES XIX. a.

Solve the equations:

1. $x+y=6$, $x-y=2$.
2. $x+y=7$, $x-y=3$.
3. $x+y=12$, $x-y=6$.
4. $x+y=16$, $x-y=0$.
5. $x-y=5$, $x+y=19$.
6. $x+y=40$, $x-y=6$.
7. $2x+3y=9$, $2x-3y=3$.
8. $3x+2y=13$, $3x-2y=5$.
9. $4x+3y=7$, $4x-3y=1$.
10. $x+2y=4$, $2x+3y=7$.
11. $2x+y=23$, $4x-y=19$.
12. $x+3y=38$, $3x-y=24$.
13. $x+5y=18$, $3x+2y=41$.
14. $7x+5y=45$, $2x-3y=4$.
15. $3y-7x=1$, $4y+x=53$.
16. $2x+7y=31$, $3x-5y=0$.
17. $2x-3y=21$, $4x+9y=117$.
18. $3x+5y=94$, $6x-7y=1$.
19. $7x+3y=10$, $35x-6y=1$.
20. $11y-6x=36$, $7y+24x=9$.
21. $14x+9y=19$, $9x+14y=4$.
22. $13x+11y=70$, $11x+13y=74$.
23. $5y-3x=85$, $12y+5x=21$.
24. $17x+9y=74$, $13x-11y=146$.

Second Method.

169. The elimination of one of the unknowns may also be effected by the method given in the following example.

Example. Solve

$$2x=5y+1 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$24-7x=3y \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Here we can eliminate x by substituting in (2) its value obtained from (1). Thus

$$2x=5y+1 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1);$$

$$\therefore\ x=\frac{1}{2}(5y+1).$$

Substituting this value of x in (2), we have

$$24-\frac{7}{2}(5y+1)=3y;$$

$$\therefore\ 48-35y-7=6y;$$

$$\therefore\ 41=41y;$$

$$\therefore\ \left.\begin{array}{l} y=1, \\ x=3. \end{array}\right\}$$

and from (1),

EXAMPLES XIX. b.

Solve the following equations by using the "Second Method":

1. $2x+y=18,\quad 3y-x=33.$

2. $3x-2y=6,\quad 6y-5x=30.$

3. $x=3y-2,\quad 9y=4x-7.$

4. $3x=7+y,\quad 5x-41=9y.$

5. $7x-2y=0,\quad 13x-5=3y.$

6. $4x+3y=0,\quad 5y+53=11x.$

7. $5x+9y=37,\quad 7y-3x=15.$

8. $3x+2y=118,\quad x+5y=191.$

9. $9x+5y=65,\quad 11x-7y=27.$

10. $17x-8y=4,\quad 6x=5y+21.$

11. $7x=3y,\quad 13+2y=9x.$

12. $3x+19=4y,\quad 7x+27=18y.$

13. $6x+15y=77,\quad 17x-y=\frac{2}{3}.$

14. $7x-5y=3,\quad 9y=2\frac{1}{5}+5x.$

15. $12x+9=y,\quad 18y-5x=56\frac{1}{2}.$

16. $5x+3y=15,\quad 11x-5y=62.$

17. $7x=10y+4,\quad 12x+18y=1.$

18. $9x-16y=45,\quad 13x-17y=10.$

Third Method.

170. The method, explained in the following example, may be sometimes used with advantage.

Example. Solve

$$4x + y = 10 \quad\ldots\ldots\ldots\ldots\ldots\ldots(1),$$
$$5x + 7y = 47 \quad\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

From (1), by transposing, we have

$$y = 10 - 4x;$$

and, from (2),

$$y = \frac{47 - 5x}{7}.$$

Equating these two values of y, we have

$$10 - 4x = \frac{47 - 5x}{7},$$

or

$$70 - 28x = 47 - 5x;$$
$$\therefore\ 23 = 23x;$$

and from (1),

$$\left.\begin{aligned} \therefore\ x &= 1, \\ y &= 6. \end{aligned}\right\}$$

EXAMPLES XIX. c.

Solve the following equations by the "Third Method":

1. $3x - y = 3,\quad 5x + 2y = 16.$
2. $x + 3y = 3,\quad 4x - 5y = 29.$
3. $3x - y = 13,\quad 5x + 3y = 3.$
4. $7x + 5y = 1,\quad 5x + 7y = 11.$
5. $4x + 9y = 47,\quad 5x - 4y = 13.$
6. $13 - 2x = 3y,\quad 23 - 8y = 7x.$
7. $9x - 11y = 15,\quad 7x - 13y = 25.$
8. $21x - 10y = 109,\quad 13x - 7y = 61.$
9. $23x - 11y = 1,\quad 15x + 7y = 29.$
10. $17x - 9y = 15,\quad 4x + 13y = 64.$
11. $9x - 25y = 61,\quad 11x + 36y = 8.$
12. $12y + 17x = 35,\quad 11y - 18x = 200.$
13. $23y - 13x = 36,\quad 25y + 11x = 14.$
14. $20y - 7x = 141,\quad 11y + 5x = 228.$
15. $13y + 14x = 59,\quad 11y + 19x = 100.$
16. $27y - 5x = 303,\quad 19y + 9x = 63.$
17. $17x + 10y = 104,\quad 5x - 24y = 300.$
18. $19y = 12x - 61,\quad 17y = 5x - 89.$

171. Any one of the methods given above will be found sufficient; but the work may sometimes be shortened by an arithmetical device.

Example. Solve $28x - 23y = 22$ ....(1),
$63x - 55y = 17$ ....(2).

Noticing that 28 and 63 contain a common factor 7, we shall make the coefficients of x in the two equations equal to the *least common multiple* of 28 and 63 if we multiply (1) by 9 and (2) by 4.

Thus $252x - 207y = 198$,
$252x - 220y = 68$;
subtracting, $13y = 130$;
that is, $y = 10$,
and therefore from (1), $x = 9$.

172. Sometimes it will be necessary to simplify the equations before applying any one of the methods of solution.

Example 1. Solve $5(x + 2y) - (3x + 11y) = 14$ ....(1),
$7x - 9y - 3(x - 4y) = 38$ ....(2).

From (1), $5x + 10y - 3x - 11y = 14$;
$\therefore\ 2x - y = 14$ ....(3).
From (2), $7x - 9y - 3x + 12y = 38$;
$\therefore\ 4x + 3y = 38$ ....(4).
From (3), $6x - 3y = 42$;
adding, $10x = 80$;
$\therefore\ x = 8$,
and from (3), $y = 2$.

Example 2. Solve $7x - {\cdot}3y = 12{\cdot}75$ ....(1),
${\cdot}2x + {\cdot}06y = 1\frac{1}{20}$ ....(2).

First clear of fractions by multiplying both equations throughout by 100. Thus we have

$700x - 30y = 1275$ ....(3),
$20x + 6y = 105$ ....(4).

Dividing (3) by 5, $140x - 6y = 255$;
adding, $160x = 360$;
$\therefore\ x = 2\frac{1}{4}$,
and from (4), $y = 10$.

173. When the value of one of the unknowns has been found, instead of substituting this value to find the value of the second unknown, it will sometimes prove easier to employ the process of Elimination.

Example. Solve $3x-\frac{y-5}{7}=\frac{4x-3}{2}$ ....(1),

$\frac{3y+4}{5}-\frac{1}{3}(2x-5)=y$ ....(2).

Clearing of fractions, we have

from (1), $42x-2y+10=28x-21$;

$\therefore\ 14x-2y=-31$ ....(3),

and from (2), $9y+12-10x+25=15y$;

$\therefore\ 10x+6y=37$ ....(4).

Eliminating y from (3) and (4), we find that

$$x=-1\tfrac{1}{13}.$$

Eliminating x from (3) and (4), we find that

$$y=7\tfrac{25}{26}.$$

EXAMPLES XIX. d.

Solve the equations :

1. $22x-9y=17$, $33x+16y=114$.
2. $14x-27y=2$, $19x-36y=4$.
3. $13x+35y=25$, $11x+28y=26$.
4. $39x+7y=107$, $52x+11y=131$.
5. $9y+28x=26$, $5y-42x=72$.
6. $19x+45y=36$, $17x+30y=63$.
7. $95y=49+23x$, $76y=102-13x$.
8. $25y-34x=227$, $9y-85x=300$.
9. $11x-45y=211$, $24x+75y=114$.
10. $3x-y=23$, $\frac{x}{3}+\frac{y}{4}=4$.
11. $2x-3y=24$, $\frac{5x}{3}-\frac{y}{2}=12$.
12. $3x-y=3$, $\frac{5x}{8}+\frac{7y}{18}=6$.
13. $12x+7y=2\frac{1}{2}$, $8y-9x=18$.
14. $\frac{4y-2}{3}=\frac{5x}{2}$, $18x-20y=3$.
15. $\frac{x-3}{5}=\frac{y-7}{2}$, $11x=13y$.
16. $7x=32+\frac{3y}{5}$, $3y=11+\frac{4x}{5}$.
17. $17y+18x=\frac{1}{3}$, $12x+13y=\frac{1}{6}$.
18. $9x+3y=\frac{3}{4}$, $27x-8y=10\frac{3}{4}$.

Solve the equations :

19. $\frac{11x-5y}{11}=\frac{3x+y}{16}$,
$8x-5y=1$.

20. $\frac{1}{5}(x-2)=\frac{1}{4}(1-y)$,
$26x+3y+4=0$.

21. $4(x-2y)-(5x+3y)=30$,
$3(3x+7y)-2(x+9y)=12$.

22. $11(2x+3y)-3(y-4x)=27$,
$6(5x-y)-(3y-14x)=19$.

23. $\frac{3x+1}{7}-\frac{2x-y}{2}=\frac{2y-x}{8}$,
$\frac{4x-2}{3}-\frac{4y-5x}{2}=\frac{x+y}{5}$.

24. $\frac{2y-25}{3}-\frac{6-x}{7}=\frac{2(y-7)}{5}$,
$\frac{29-x}{8}-\frac{3y-1}{10}=\frac{4-x}{3}$.

25. $\frac{5x-18}{4}=\frac{3x+2y}{12}=\frac{5y-4x}{7}$.

26. $\frac{3x-4y}{24}=\frac{4x-3y}{11}=1-x+y$.

27. $3\cdot 75x-1\cdot 5y=27$,
$7x+6y=68$.

28. $2x+\cdot 4y=1\cdot 2$,
$3\cdot 4x-\cdot 02y=\cdot 01$.

29. $3x+\cdot\dot{3}y-\cdot 5(x-y)=35$,
$\cdot 75y-\frac{x}{2}+\cdot\dot{6}(3x-2y)=11\frac{1}{4}$.

30. $\cdot 1\dot{6}x+\cdot 05y-\cdot 1=1\frac{11}{12}$,
$\cdot 4y+\cdot 08\dot{3}x-\cdot 4=3\frac{7}{30}$.

174. Definition. If the product of two quantities is equal to unity, each quantity is said to be the **reciprocal** of the other. Thus the following pairs are *reciprocals* :

$$3 \text{ and } \frac{1}{3}; \quad x \text{ and } \frac{1}{x}; \quad \frac{a}{b} \text{ and } \frac{b}{a}.$$

Simultaneous equations may often be conveniently solved by taking the reciprocals of x and y, that is $\frac{1}{x}$ and $\frac{1}{y}$, as the unknown quantities.

Example. Solve
$$\frac{8}{x}-\frac{9}{y}=1 \quad\text{..............................(1)};$$
$$\frac{10}{x}+\frac{6}{y}=7 \quad\text{..............................(2)}.$$

Multiplying (1) by 2, and (2) by 3, we have
$$\frac{16}{x}-\frac{18}{y}=2,$$
$$\frac{30}{x}+\frac{18}{y}=21;$$

adding,
$$\frac{46}{x}=23;$$

multiplying up,
$$46=23x;$$

and substituting in (1),
$$\therefore \left.\begin{aligned} x&=2,\\ y&=3.\end{aligned}\right\}$$

EXAMPLES XIX. e.

Solve the equations :

1. $\dfrac{1}{x}+\dfrac{1}{y}=10,\quad \dfrac{1}{x}-\dfrac{1}{y}=6.$

2. $\dfrac{1}{x}+\dfrac{1}{y}=27,\quad \dfrac{1}{x}-\dfrac{1}{y}=13.$

3. $\dfrac{1}{x}-\dfrac{1}{y}=\dfrac{1}{2},\quad \dfrac{2}{x}+\dfrac{6}{y}=5.$

$x = 1$
$y = 2$

4. $\dfrac{3}{x}+\dfrac{6}{y}=4,\quad \dfrac{9}{x}-\dfrac{2}{y}=2.$

5. $\dfrac{5}{x}+\dfrac{4}{y}=58,\quad \dfrac{3}{x}+\dfrac{7}{y}=67.$

6. $\dfrac{9}{x}-\dfrac{4}{y}=8,\quad \dfrac{13}{x}+\dfrac{7}{y}=101.$

7. $\dfrac{15}{x}-\dfrac{1}{y}=4\tfrac{1}{2},\quad \dfrac{9}{x}+\dfrac{2}{y}=4.$

8. $\dfrac{2}{x}+\dfrac{5}{y}=\dfrac{5}{6},\quad \dfrac{3}{x}+\dfrac{4}{y}=\dfrac{9}{10}.$

9. $\dfrac{2}{x}-\dfrac{3}{2y}=\dfrac{41}{35},\quad \dfrac{2\frac{1}{2}}{2x}+\dfrac{3\frac{1}{2}}{y}=-\dfrac{73}{70}.$

10. $\dfrac{2}{x}+3y=15,\quad \dfrac{5}{x}-4y=3.$

11. $\dfrac{1}{x}+2y=1\tfrac{1}{4},\quad \dfrac{2}{x}+y=1.$

12. $\dfrac{3}{x}-2y=1\tfrac{3}{8},\quad \dfrac{16}{x}+3y=\dfrac{1}{2}.$

175. An easy extension of the foregoing methods will enable us to solve simultaneous equations containing three unknown quantities. It will be found that in order to solve simultaneous equations which contain *three unknowns* we must have *three equations.* The method of procedure is conveniently embodied in the following rule :

Rule. *Eliminate one of the unknowns from any pair of the equations, and then eliminate the same unknown from another pair. Two equations involving two unknowns are thus obtained, which may be solved by the rules already given. The remaining unknown is then found by substituting in any one of the given equations.*

Example. Solve

$$7x+5y-7z=-8 \quad\ldots\ldots\ldots\ldots(1),$$
$$4x+2y-3z=0 \quad\ldots\ldots\ldots\ldots(2),$$
$$5x-4y+4z=35 \quad\ldots\ldots\ldots\ldots(3).$$

Choose y as the unknown to be eliminated.

Multiplying (2) by 5, $20x+10y-15z=0$;
multiplying (1) by 2, $14x+10y-14z=-16$;
subtracting, $6x-z=16 \quad\ldots\ldots\ldots\ldots(4).$

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Multiplying (2) by 2, $8x+4y-6z=0$;
from (3), $5x-4y+4z=35$;
adding, $13x-2z=35$ ……………………(5).

Multiplying (4) by 2, $12x-2z=32$;
subtracting, $x=3$.

Substituting in (4), $z=2$,
substituting in (2), $y=-3$.

EXAMPLES XIX. f.

Solve the equations:

1. $x+y+z=7,$
$2x+3y-z=0,$
$3x+4y+2z=17.$

2. $x+y-z=8,$
$4x-y+3z=26,$
$2x+y-4z=8.$

3. $2x+y+z=8,$
$5x-3y+2z=3,$
$7x+y+3z=20.$

4. $3x+y-z=3,$
$2x-y+3z=20,$
$7x+y+z=23.$

5. $5x-4y+z=3,$
$3x+y-2z=31,$
$x+4y+z=15.$

6. $4x-5y+6z=3,$
$8x-7y-3z=9,$
$7x-8y+9z=6.$

7. $5z-3x=4(1+y),$
$2(x+2z)=8+3y,$
$2y+3z=14-x.$

8. $x+z=2y,$
$9x+3z=8y,$
$2x+3y+5z=36.$

9. $\frac{1}{4}x+\frac{1}{6}y+\frac{1}{3}z=8,$
$\frac{1}{2}x-\frac{1}{9}y+\frac{1}{6}z=5,$
$\frac{1}{3}x+\frac{1}{2}y-z=7.$

10. $\frac{1}{2}x+\frac{1}{3}y+\frac{1}{6}z=12,$
$\frac{1}{2}y+\frac{1}{3}z-\frac{1}{6}x=8,$
$\frac{1}{3}z+\frac{1}{2}x=10.$

11. $y+z-x=z+x-3y=\frac{1}{2}(x+y-2z)=1.$

12. $x+z-1=\frac{1}{2}(x+4z-8)=\frac{1}{3}(x+9z-27)=y.$

Simultaneous Equations with Literal Coefficients.

176. The following examples will be readily understood provided the pupil bears in mind that the literal coefficients represent *known* quantities, and must, therefore, appear in the solution.

Example 1. Solve the simultaneous equations:

$$ax - by = c \quad \text{.......... (1)},$$

$$px + qy = r \quad \text{.......... (2)}.$$

Multiplying (1) by q and (2) by b, to eliminate y, we have

$$aqx - bqy = cq,$$

$$bpx + bqy = br;$$

adding, $$(aq + bp)x = cq + br;$$

$$\therefore \quad x = \frac{cq + br}{aq + bp}.$$

We might obtain y by substituting this value of x in *either* of the equations (1) and (2); but y is more conveniently found by eliminating x, as follows.

Multiplying (1) by p and (2) by a, we have

$$apx - bpy = cp,$$

$$apx + aqy = ar;$$

subtracting, $$(aq + bp)y = ar - cp;$$

$$\therefore \quad y = \frac{ar - cp}{aq + bp}.$$

Example 2. Solve $a(x + y) - b(x - y) = 2a$ .... (1),

$$(a^2 - b^2)(x - y) = 4ab \quad \text{.......... (2)}.$$

From (1), $$ax + ay - bx + by = 2a;$$

$$\therefore \quad x(a - b) + y(a + b) = 2a \quad \text{.......... (3)}.$$

From (2), $$x(a^2 - b^2) - y(a^2 - b^2) = 4ab \quad \text{.......... (4)}.$$

Multiplying (3) by $a - b$, we have

$$x(a - b)^2 + y(a^2 - b^2) = 2a^2 - 2ab.$$

From (4), $$x(a^2 - b^2) - y(a^2 - b^2) = 4ab;$$

adding, $$x(a^2 - 2ab + b^2 + a^2 - b^2) = 2a^2 + 2ab,$$

or $$x(2a^2 - 2ab) = 2a(a + b);$$

$$\therefore \quad x = \frac{2a(a + b)}{2a(a - b)} = \frac{a + b}{a - b}.$$

Substituting in (3), $\frac{a+b}{a-b}(a-b)+y(a+b)=2a,$

or $y(a+b)=2a-(a+b)\,;$

$$\therefore\ y=\frac{a-b}{a+b}.$$

EXAMPLES XIX. g.

Solve the equations :

1. $ax+by=2ab,$
   $bx-ay=b^2-a^2.$

2. $2bx-ay=ab,$
   $bx+2ay=3ab.$

3. $a(x-a)=b(y-a),$
   $b(x+b)=a(y+b).$

4. $ax+by=a^2+ab,$
   $bx+ay=ab+b^2.$

5. $bx-ay=b^2,$
   $(a-b)x+by=a^2.$

6. $qx-py+q^2=0,$
   $(p+q)x+qy=p^2.$

7. $\frac{x}{a}+\frac{y}{b}=2,$
   $ax-by=a^2-b^2.$

8. $\frac{x}{2a}+\frac{y}{2b}=\frac{1}{a+b},$
   $ax-by=a-b.$

9. $a(x+y)+b(x-y)=2ab,$
   $a(x+y)-b(x-y)=ab.$

10. $(a^2+b^2)(x-1)=ab(2x-y),$
    $4x=y+2.$

11. $\frac{a-b}{x}+\frac{a+b}{y}=\frac{2(a^2+b^2)}{a^2-b^2},$
    $\frac{a+b}{x}+\frac{a-b}{y}=2.$

12. $\frac{x}{a-b}+\frac{y}{a+b}=\frac{8ab}{a^2-b^2},$
    $\frac{x}{a+3b}-\frac{y}{a-3b}=2.$

13. $\frac{x}{c}+\frac{y}{c+d}=c,$
    $\frac{x}{c-d}-\frac{y}{d}=-d.$

14. $\frac{x}{a}+\frac{y}{b}=2a,$
    $\frac{x}{a-b}+\frac{y}{a+b}=a+b.$

15. $\frac{x-a}{2}+\frac{y-b}{3}=a,$
    $\frac{x-b}{3}+\frac{y-a}{2}=b.$

16. $\frac{m}{x}+\frac{n}{y}=a,$
    $\frac{n}{x}+\frac{m}{y}=b.$

17. $\frac{p}{x}+\frac{q}{y}=0,$
    $px+qy=r.$

18. $lx=my,$
    $\frac{a}{x}-\frac{b}{y}=c.$

CHAPTER XX.

PROBLEMS LEADING TO SIMULTANEOUS EQUATIONS.

177. IN the last chapter we have seen that there must always be as many equations as there are unknown quantities whose values are to be found. Consequently, in problems which may be solved by using simultaneous equations, there must be as many independent conditions, each of which can be stated in the form of a separate equation, as there are quantities to be determined.

Example I. Three-fourths of the sum of two numbers is 57, and if the greater be subtracted from three times the less the remainder is 40; find the numbers.

Let x be the greater number, and y the less.

From the first condition, $\frac{3}{4}(x+y)=57$,

or $3x+3y=228$ ....(1).

From the second, $3y-x=40$ ....(2).

Subtracting, we obtain $4x=188$;

$\therefore\ x=47$;

and from (2), $y=29$.

Thus the numbers are 47 and 29.

Example II. If unity be subtracted from the numerator of a certain fraction, and 2 added to the denominator, it reduces to $\frac{2}{3}$; if 3 be added to the numerator, and the denominator be multiplied by 2, it reduces to $\frac{3}{5}$. What is the fraction?

Let x be the numerator of the fraction, and y the denominator; then the fraction is $\frac{x}{y}$.

From the first supposition,

$$\frac{x-1}{y+2}=\frac{2}{3} \text{ ....(1),}$$

from the second,

$$\frac{x+3}{2y}=\frac{3}{5} \text{ ....(2).}$$

Multiplying up, we have from (1),

$$3x-3=2y+4,$$

or $3x-2y=7$ ....(3),

and from (2), $5x+15=6y,$

or $5x-6y=-15$ ....(4).

Multiplying (3) by 3, $9x-6y=21;$

subtracting, $4x=36;$

$$\therefore\ x=9,$$

and from (3), $y=10.$

Therefore the fraction is $\frac{9}{10}$.

EXAMPLES XX. a.

1. Find two numbers whose sum is 25, and whose difference is 7.

2. Find two numbers whose sum is 61, and whose difference is 15.

3. One-fourth of the sum of two numbers is 13, and one-sixth of their difference is 3; find them.

4. One-seventh of the sum of two numbers is 6, and four times their difference is 64; find them.

5. Find two numbers which are such that twice the greater exceeds three times the less by 10; and such that one-fifth of the greater is less than 20 by one-third of the less.

6. Find two numbers such that one-third of the greater exceeds one-half of the less by 1, and one-fifth of the greater added to one-sixth of the less equals one-half of the less.

7. The difference of two numbers is five-sixths of their sum, and the greater exceeds 10 times the less by 3; find the numbers.

8. One man said to another: "If you give me half your money I shall have £5." The other replied "I shall have £5 if you give me a third of your money." How much had each?

9. Find a fraction which becomes equal to $\frac{1}{2}$ if 1 be subtracted from both numerator and denominator, and equal to $\frac{2}{3}$ if 1 be added to both numerator and denominator.

10. Find a fraction such that, if its numerator be diminished by unity, it reduces to $\frac{1}{2}$, and becomes equal to 2 when the numerator is increased by 5 and the denominator diminished by 6.

11. If 4 be taken from the denominator of a certain fraction it reduces to 1; if the numerator be multiplied by 3, and the denominator increased by 5 it reduces to $\frac{3}{2}$. What is the fraction?

12. Add 1 to the numerator and denominator of a certain fraction and it reduces to $\frac{4}{5}$; subtract 5 from each, and it reduces to $\frac{1}{2}$: required the fraction.

Example III. Five cows and nine sheep are worth £102, while 6 cows and 7 sheep are worth £111; find the value of a cow and a sheep respectively.

Suppose a cow to cost x pounds,
and a sheep ,, y ,,

Then from the question we have

$$5x+9y=102 \quad\text{..............................(1)},$$
$$6x+7y=111 \quad\text{..............................(2)}.$$

Multiplying (1) by 6, and (2) by 5, we have

$$30x+54y=612,$$
$$30x+35y=555.$$

Subtracting, $$19y=57;$$
$$\therefore\ y=3.$$

Substituting in (2) $$6x+21=111;$$
$$\therefore\ x=15.$$

$\therefore$ the cost of a cow is £15, and the cost of a sheep £3.

Example IV. A certain number of persons paid a bill; if there had been 12 fewer each would have paid 1*s.* 6*d.* more; if there had been 8 more each would have paid 6*d.* less: find the number of persons and what each had to pay.

Let x be the number of persons, and y the number of shillings paid by each.

Then the total number of shillings paid is $x\,.\,y$.

If $(x-12)$ persons paid $(y+1\frac{1}{2})$ shillings each,

the total number of shillings paid $=(x-12)(y+1\frac{1}{2})$;

and if $(x+8)$ persons paid $(y-\frac{1}{2})$ shillings each,

the total number of shillings paid $=(x+8)(y-\frac{1}{2})$.

$$\therefore\ xy=(x-12)(y+1\tfrac{1}{2}) \quad\text{........................(1)},$$
and $$xy=(x+8)(y-\tfrac{1}{2}) \quad\text{..........................(2)}.$$

From (1), $$xy=xy+1\tfrac{1}{2}x-12y-18;$$
that is, $$3x-24y=36,$$
or $$x-8y=12 \quad\text{..................................(3)}.$$

From (2), $$xy=xy-\frac{x}{2}+8y-4;$$
that is, $$x-16y=-8 \quad\text{..................................(4)}.$$

By combining (3) and (4) we find that $x=32$, $y=2\frac{1}{2}$.

Thus there were 32 persons and each paid 2*s.* 6*d.*

13. A horse and a cow are together worth £42; while 4 horses and 7 cows cost £213: find the price of each animal.

14. If either 9 horses and 7 cows, or 6 horses and 13 cows, can be bought for £300, what is the price of each?

15. A man sells 15 animals, consisting of horses and sheep, for £275. If the price of a horse is £45 and of a sheep £5, how many of each did he sell?

16. If 10 lbs. of tea and 8 lbs. of coffee cost £1. 16*s.* 6*d.*, while 6 lbs. of tea and 5 lbs. of coffee cost £1. 2*s.* 3*d.*; find the price per pound of tea and coffee.

17. The wages of 24 men and 16 boys amount to £5. 16*s.* per day; half that number of men, with 21 more boys, would earn the same money. What are the daily wages of each man and boy?

18. I buy two pieces of cloth for £25. 6*s.*, one piece being 16*s.* and the other 18*s.* per yard. I sell them at a profit of 2*s.* per yard, and gain on the whole £3. How long was each piece?

19. A certain number of persons paid a bill; if there had been 10 more, each would have paid 1*s.* less; if there had been 5 fewer, each would have paid 1*s.* more: find the number of persons, and what each had to pay.

20. A certain subscription is raised in a boys' school; had each boy given a penny less, the money would have been obtained from 20 more, and if each had given 2*d.* more, from 25 fewer subscribers. How many contributors were there, and what did each give?

21. A fruiterer packs a certain number of oranges into some boxes; if he could put 20 more oranges into each box, he would require 3 boxes less, and if 5 less into each box, 1 box more: how many oranges does he pack, and how many boxes does he use?

22. If the breadth of a certain rectangle were increased by 5 yards, and its length diminished by 10 yards, its area would be increased by 200 square yards; whilst if its breadth were diminished by 5 yards, and its length increased by 15 yards, its area would be lessened by 75 square yards. What are its length and breadth?

23. A sum of money is to be divided equally among a certain number of boys. If there were 3 boys less, each would receive exactly 2*s.*, and if there were 2 more, each would get only 1*s.* 6*d.* How much money is there for division, and how many boys are to share it?

24. A sum of money is to be equally divided among a certain number of persons. If there had been two more persons, each would have received exactly 2*s.*, and if there had been one less, each would have received 2*s.* 6*d.* Among how many persons was the money divided?

178. In a problem involving the digits of a number the pupil should carefully notice the way in which the value of a number is algebraically expressed in terms of its digits.

Consider a number of three digits such as 435. The *sum of the digits* is $4+3+5$, but the place value of the digit 4 is 4×100, and that of the digit 3 is 3×10. Thus the *value of the number* is $4\times100+3\times10+5$. Similarly a number whose digits, beginning from the left, are x, y, z

$$=x \text{ hundreds}+y \text{ tens}+z \text{ units}$$
$$=100x+10y+z,$$

whilst the sum of its digits $=x+y+z$.

Example. A certain number of two digits is three times the sum of its digits, and if 45 be added to it the digits will be reversed; find the number.

Let x be the digit in the tens' place, y the digit in the units place; then the number will be represented by $10x+y$, and the number formed by reversing the digits will be represented by $10y+x$.

The sum of the digits is $x+y$.

Hence we have the two equations

$$10x+y=3(x+y)\ldots\ldots\ldots\ldots(1),$$

and

$$10x+y+45=10y+x\ldots\ldots\ldots\ldots(2).$$

From (1), $7x=2y$;

from (2), $y-x=5$.

From these equations we obtain $x=2$, $y=7$.

Thus the number is 27.

EXAMPLES XX. b.

1. A number of two digits is such that if 19 be added to it the digits will be reversed; if the sum of the digits is 7, find the number.

2. A number of two digits is equal to eight times the sum of its digits; if 45 be subtracted from the number, the digits will be reversed: find the number. 72

3. A number of two digits has its digits reversed if 18 is taken from it; the sum of the digits is 12; find the number.

4. A certain number of two digits is two-ninths of what it would be if the digits were reversed. If the number is increased by the sum of its digits the result is 27; find the number.

5. A number of two digits exceeds four times the sum of its digits by 3; if the number be increased by 18, the result is the same as if the number formed by reversing the digits were diminished by 18. Find the number.

6. Two numbers are formed by the same two digits, and if the lesser number be divided by the greater the quotient is $\frac{4}{7}$, and if the lesser be subtracted from the greater the remainder is 27. Find the numbers.

7. A certain number consists of two digits. If 5 be added to the number, and the result divided by the units' digit, the quotient is 6; and if 10 be subtracted from the number, and the remainder divided by the sum of the digits, the quotient is 3. What is the number?

8. Of a number consisting of two digits the units' digit is the greater. If the number is increased by 3, and the result divided by the difference of the digits, the quotient is 30; and if it is diminished by 48, the remainder is equal to three-fourths of the sum of the digits. Find the number.

179. The following examples contain some special features, and should be carefully studied by the pupil before he works through the miscellaneous problems of the next exercise.

Example I. I spend 3*s.* in buying apples at 4 a penny and oranges at 3 a penny, and then dispose of three-fourths of my apples and half of my oranges for 2*s.*, which was a penny more than they cost me; how many of each did I buy?

Let x be the number of apples and y the number of oranges.

$$x \text{ apples cost } \frac{x}{4} \text{ pence,}$$

$$\text{and } y \text{ oranges cost } \frac{y}{3} \quad ,,$$

hence $$\frac{x}{4}+\frac{y}{3}=36,$$

or $$3x+4y=432 \quad \text{.......................(1).}$$

$$\frac{3}{4}x \text{ apples cost } \frac{3}{4}\cdot\frac{x}{4}, \text{ or } \frac{3x}{16} \text{ pence,}$$

$$\text{and } \frac{1}{2}y \text{ oranges cost } \frac{1}{2}\cdot\frac{y}{3}, \text{ or } \frac{y}{6} \text{ pence.}$$

But, by the question, these together cost 1*s.* 11*d.*, hence

$$\frac{3x}{16}+\frac{y}{6}=23,$$

or $$9x+8y=1104 \quad \text{.......................(2).}$$

By combining equations (1) and (2) we obtain $x=80$, $y=48$.

Thus there were 80 apples and 48 oranges.

Example II. Two persons, 27 miles apart, setting out at the same time are together in 9 hours if they walk in the same direction, but in 3 hours if they walk in opposite directions; find their rates of walking.

Suppose the faster walker goes x miles per hour,
and ,, slower ,, ,, y ,, ,, .

When they walk in the *same* direction the faster walker gains on the other $(x-y)$ miles per hour; and in 9 hours he will gain $9(x-y)$ miles.

Therefore $$9(x-y)=27,$$
or $$x-y=3 \quad \ldots\ldots\ldots\ldots(1).$$

When they walk in *opposite* directions they lessen the distance between them by $(x+y)$ miles per hour, and in three hours this decrease is $3(x+y)$ miles.

Therefore $$3(x+y)=27,$$
or $$x+y=9 \quad \ldots\ldots\ldots\ldots(2).$$
From (1) $$x-y=3;$$
whence we find $x=6$, $y=3$.

Thus the rates of walking are 6 and 3 miles per hour respectively.

Example III. A grocer buys 15 lbs. of figs and 28 lbs. of currants for £1. 1*s*. 8*d*.; by selling the figs at a loss of 10 per cent., and the currants at a gain of 30 per cent., he clears 2*s*. 6*d*. on his outlay; how much per pound did he pay for each?

Let x, y denote the number of pence in the price of a pound of figs and currants respectively; then the outlay is
$$(15x+28y) \text{ pence.}$$
Therefore $$15x+28y=260 \quad \ldots\ldots\ldots\ldots(1).$$

The loss upon the figs is $\frac{1}{10}\times 15x$ pence, and the gain upon the currants is $\frac{3}{10}\times 28y$ pence; therefore the net gain is
$$\left(\frac{42y}{5}-\frac{3x}{2}\right) \text{ pence};$$
$$\therefore \quad \frac{42y}{5}-\frac{3x}{2}=30;$$
that is, $$28y-5x=100 \quad \ldots\ldots\ldots\ldots(2).$$

From (1) and (2) we find that $x=8$, and $y=5$, that is the figs cost 8*d*. a pound, and the currants cost 5*d*. a pound.

EXAMPLES XX. c.

1. I spend 3*s.* 4*d.* in buying eggs at 2 for 1*d.*, and apples at 3 for 1*d.* If I were to sell them all alike at the rate of 20 for 1*s.*, I should gain 1*s.* 2*d.* How many of each did I buy?

2. By purchasing pen-holders at 8*d.* a score, and lead pencils at 9*d.* a doz. at a total outlay of 5*s.* 5*d.*, and selling them all at a uniform price of 11 for 8*d.*, I gain 1*s.* 3*d.* How many of each did I buy?

3. A man sold one sort of oranges at 5 for 2*d.*, and another sort at 16 for 1*s.* If he had sold them all at a half-penny each, he would have received 2*d.* less; if he had sold all at 3 for 2*d.*, he would have received 1*s.* 6*d.* more. How many of each sort did he sell?

4. On Monday a hawker disposes of the whole of his stock of boot-laces at 6 for $2\frac{1}{2}d.$, and four-ninths of his stock of buttons at 6 for $1\frac{1}{2}d.$, the proceeds amounting to 3*s.* 6*d.*; and on Tuesday he sells the remainder of the buttons at the same price, for 1*s.* 3*d.* less than he received for the boot-laces: with how many of each did he start out on Monday?

5. At 9 a.m. a man starts from *A* and walks continuously at the rate of $3\frac{1}{2}$ miles per hour to meet his friend, who starts at the same hour from *B*, 56 miles away. If the latter walks at the rate of 4 miles per hour, but stops an hour by the way, when and where do they meet?

6. *A*, *B* and *C* travel from the same place at the rates of 5, 6, and 8 miles an hour respectively; if *B* starts 2 hours after *A*, how long after *B* must *C* start in order that they may overtake *A* at the same instant?

7. A boat goes up-stream 30 miles and then down-stream 44 miles in 10 hours, and it also goes up-stream 40 miles and down-stream 55 miles in 13 hours; find the rate of the stream and of the boat.

8. Find the distance between two towns when by increasing the speed 7 miles per hour a train can perform the journey in 1 hour less, and by reducing the speed 5 miles per hour can perform the journey in 1 hour more.

9. A train travelled a certain distance at a uniform rate. Had the speed been 9 miles an hour more, the journey would have occupied 3 hours less; and had the speed been 6 miles an hour less the time taken would have been 3 hours more. Find the distance.

10. A person buys 20 yards of cloth and 25 yards of canvas for £1. 17*s.* 6*d.* By selling the cloth at a gain of 15 per cent. and the canvas at a gain of 20 per cent. he clears 6*s.* 3*d.*; find the price of each per yard.

11. A dealer spends £760 in buying horses at £24 each and cows at £20 each; through disease he loses 20 per cent. of the horses and 15 per cent. of the cows. By selling the remaining animals at cost price he receives £628; find how many of each he bought.

12. An income of £160 is derived partly from money invested at $3\frac{1}{2}$ per cent. per annum, and partly from money invested at 3 per cent. per annum; if the investments were interchanged the income would be £165. How much is invested at each rate?

13. A boy has 6*s.* with which he is to buy two kinds of note books. He finds that if he asks for 11 of the smaller size and 13 of the larger he will require 2*d.* more; if he asks for 13 of the smaller and 11 of the larger he will have 2*d.* over. Find the price of each kind.

14. A certain sum of money is divided among *A*, *B*, and *C*. *B*'s share is sixpence more than half the sum of the shares of *A* and *C*. *A*'s share is four shillings less than half the sum of the shares of *B* and *C*. If the shares of *A* and *B* together amount to 33*s.*, find how much each receives.

15. At a certain election there were two rival candidates, and their supporters were conveyed to the polling-booths in carriages capable of accommodating 8 and 12 voters respectively. If the voters, 740 in all, just filled 75 carriages, find by what majority the election was won.

16. At a flower show, at which 1250 attended, outsiders were charged 1*s.*, villagers 6*d.*, and school children 1*d.*, and the total receipts were £35. There were three times as many villagers as outsiders; how many outsiders came?

17. A bag contained shillings and half-crowns amounting to £5. Half of the shillings were taken out and replaced by half-crowns. If the value of the contents was then £6. 17*s.* 6*d.*, how many shillings did the bag contain at first?

18. By selling 6 horses and buying 8 cows a dealer increases his cash by £70. He then, at the same prices, buys 7 horses and sells 12 cows, and thereby decreases his cash by £43. Find the price of each horse and cow.

19. There is a number whose three digits, from left to right, are in descending order of magnitude and differ from each other in succession by the same amount. If the number be divided by the sum of its digits the quotient is 48; and if from the number 198 be subtracted, the digits of the difference are the same as in the original number, but in reverse order; find the number.

20. A train running from *A* to *B* meets with an accident 50 miles from *A*, after which it travels with three-fifths of its original velocity and arrives 3 hours late at *B*; if the accident had occurred 50 miles further on, it would have been only 2 hours late. Find the distance from *A* to *B* and the original velocity of the train.

CHAPTER XXI.

COMPLEX FRACTIONS.

180. DEFINITION. A fraction which contains a fractional expression in its numerator or denominator, or in both of them, is called a **Complex Fraction.**

Thus $\dfrac{a}{\dfrac{b}{c}}$, $\dfrac{\dfrac{a}{b}}{x}$, $\dfrac{\dfrac{a}{b}}{\dfrac{c}{d}}$, $\dfrac{y}{1-\dfrac{1}{x}}$ are Complex Fractions.

181. An algebraical fraction has been defined in Art. 135 as the result obtained by *dividing the numerator by the denominator.*

Thus
$$\frac{\dfrac{a}{b}}{\dfrac{c}{d}}=\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}.$$

The following particular cases should be carefully noticed so that the results may be written down by inspection.

$$\frac{1}{\dfrac{a}{b}}=1\div\frac{a}{b}=1\times\frac{b}{a}=\frac{b}{a}. \qquad \frac{a}{\dfrac{1}{b}}=a\div\frac{1}{b}=a\times\frac{b}{1}=ab.$$

Similarly
$$\frac{\dfrac{1}{a-x}}{\dfrac{1}{a+x}}=\frac{a+x}{a-x}.$$

182. The following examples illustrate the simplification of complex fractions.

Example 1. Simplify $\dfrac{x+\dfrac{a^2}{x}}{x-\dfrac{a^4}{x^3}}$.

$$\text{The expression}=\left(x+\frac{a^2}{x}\right)\div\left(x-\frac{a^4}{x^3}\right)=\frac{x^2+a^2}{x}\div\frac{x^4-a^4}{x^3}$$
$$=\frac{x^2+a^2}{x}\times\frac{x^3}{x^4-a^4}=\frac{x^2}{x^2-a^2}.$$

Example 2. Simplify $\dfrac{\dfrac{3}{a}+\dfrac{a}{3}-2}{\dfrac{a}{6}+\dfrac{1}{2}-\dfrac{3}{a}}$.

Here the reduction may be simply effected by multiplying the fractions above and below by $6a$, which is the L.C.M. of the denominators.

$$\text{Thus the expression}=\frac{18+2a^2-12a}{a^2+3a-18}$$

$$=\frac{2(a^2-6a+9)}{(a+6)(a-3)}$$

$$=\frac{2(a-3)}{a+6}.$$

Note. Since the numerator and denominator of the complex fraction were *both* multiplied by $6a$, its value of course was unaltered by that operation.

Example 3. Simplify $\dfrac{\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{a^2-b^2}{a^2+b^2}}{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}$.

$$\text{The numerator}=\frac{(a^2+b^2)^2-(a^2-b^2)^2}{(a^2+b^2)(a^2-b^2)}$$

$$=\frac{4a^2b^2}{(a^2+b^2)(a^2-b^2)}.$$

$$\text{Similarly the denominator}=\frac{4ab}{(a+b)(a-b)}.$$

$$\text{Hence the fraction}=\frac{4a^2b^2}{(a^2+b^2)(a^2-b^2)}\div\frac{4ab}{(a+b)(a-b)}$$

$$=\frac{4a^2b^2}{(a^2+b^2)(a^2-b^2)}\times\frac{(a+b)(a-b)}{4ab}$$

$$=\frac{ab}{a^2+b^2}.$$

Note. To ensure accuracy and neatness, when the numerator and denominator are somewhat complicated, the beginner is advised to simplify each separately, as in the above example.

EXAMPLES XXI. a.

Find the value of

1. $\dfrac{1}{a+\dfrac{b}{c}}$.
2. $\dfrac{1}{\dfrac{x}{y}-z}$.
3. $\dfrac{x+1}{1-\dfrac{1}{x^2}}$.
4. $\dfrac{c}{\dfrac{1}{1-d}}$.
5. $\dfrac{\dfrac{1}{b}+\dfrac{1}{a}}{\dfrac{a}{b}-\dfrac{b}{a}}$.
6. $\dfrac{1+\dfrac{x}{a}}{\dfrac{x^2}{a}-a}$.
7. $\dfrac{\dfrac{3}{a}-\dfrac{a}{3}}{\dfrac{1}{a}-\dfrac{1}{3}}$.
8. $\dfrac{\dfrac{2x}{y}+\dfrac{5}{3}}{\dfrac{5y}{2x}+3}$.
9. $\dfrac{4-\dfrac{8b}{3a}}{\dfrac{3a}{b}-2}$.
10. $\dfrac{\dfrac{a^2c}{b}-\dfrac{b}{c}}{a+\dfrac{b}{c}}$.
11. $\dfrac{\dfrac{x}{y}+\dfrac{a}{b}}{\dfrac{x}{b}+\dfrac{a}{y}}$.
12. $\dfrac{\dfrac{x}{a}-\dfrac{y}{b}}{\dfrac{b}{y}-\dfrac{a}{x}}$.
13. $\dfrac{\dfrac{1}{a}-\dfrac{2}{a^2}+1}{1+\dfrac{4}{a}-\dfrac{5}{a^2}}$.
14. $\dfrac{\dfrac{6}{x}-5+x}{\dfrac{1}{2}+\dfrac{1}{2x}-\dfrac{6}{x^2}}$.
15. $\dfrac{\dfrac{1}{a}+\dfrac{2}{a^2}-\dfrac{15}{a^3}}{a-\dfrac{25}{a}}$.
16. $\dfrac{x-3-\dfrac{30}{x-2}}{x-1-\dfrac{20}{x-2}}$.
17. $\dfrac{\dfrac{m}{1+m}+\dfrac{1-m}{m}}{\dfrac{m}{1+m}-\dfrac{1-m}{m}}$.
18. $\dfrac{\dfrac{a}{a+b}+\dfrac{b}{a-b}}{\dfrac{a}{a-b}-\dfrac{b}{a+b}}$.

183. The fraction in the following example is called a **Continued Fraction**, and in simplifying cases of this kind we begin from the lowest fraction, and simplify step by step.

Example. Find the value of $\dfrac{1}{4-\dfrac{3}{2+\dfrac{x}{1-x}}}$.

The expression $=\dfrac{1}{4-\dfrac{3}{\dfrac{2-2x+x}{1-x}}}=\dfrac{1}{4-\dfrac{3(1-x)}{2-x}}$

$$=\frac{1}{\dfrac{8-4x-3+3x}{2-x}}=\frac{1}{\dfrac{5-x}{2-x}}$$

$$=\frac{2-x}{5-x}.$$

EXAMPLES XXI. b.

Find the value of

1. $1-\dfrac{1}{1+\dfrac{1}{a}}$.

2. $2-\dfrac{1}{1-\dfrac{1}{x}}$.

3. $a+\dfrac{a}{1-\dfrac{1}{a}}$.

4. $1-\dfrac{a-b}{a+b+\dfrac{b^2}{a-b}}$.

5. $a-\dfrac{a-2b}{2-\dfrac{a+b}{a-b}}$.

6. $x-\dfrac{y}{1+\dfrac{1}{1+\dfrac{y}{x}}}$.

7. $a-\dfrac{1}{c+\dfrac{1}{c+\dfrac{1}{a}}}$.

8. $1-\dfrac{x+y}{x+y-\dfrac{x+\dfrac{y^2}{x}}{1-\dfrac{y}{x}}}$.

9. $\dfrac{1}{x+\dfrac{1}{1+\dfrac{x+1}{3-x}}}$.

10. $\dfrac{\dfrac{2}{b+c}-\dfrac{1}{b}}{c+\dfrac{bc}{c-2b}}+\dfrac{\dfrac{2}{b+c}-\dfrac{1}{c}}{b+\dfrac{bc}{b-2c}}$.

11. $\dfrac{\dfrac{2yz}{y+z}-y}{\dfrac{1}{z}+\dfrac{1}{y-2z}}+\dfrac{\dfrac{2yz}{y+z}-z}{\dfrac{1}{y}+\dfrac{1}{z-2y}}$.

12. $2-\dfrac{3}{1+\dfrac{5a}{2+\dfrac{4a^2-1}{a+1}}}$.

13. $2-\dfrac{2}{1-\dfrac{3}{2-\dfrac{3}{1-x}}}$.

14. $\dfrac{y^3+x^3}{y-\dfrac{x}{1+\dfrac{x}{y-x}}}-\dfrac{y^3-x^3}{y+\dfrac{x}{1-\dfrac{x}{y+x}}}$.

184. The following exercise contains miscellaneous examples which illustrate most of the processes connected with fractions.

EXAMPLES XXI. c.

Simplify the following fractions:

1. $\dfrac{2}{4x^2-1}+\dfrac{1}{(2x+1)^2}$.

2. $\dfrac{x^2+5x+4}{x^3+4x^2+5x+2}$.

3. $\dfrac{x^4-y^4}{a^2b+ab^2}\times\dfrac{a+b}{(x+y)^2}\div\dfrac{(x-y)^2}{ab}$.

4. $\dfrac{5}{2(x+1)}-\dfrac{1}{10(x-1)}-\dfrac{24}{5(2x+3)}$.

Simplify the following fractions:

5. $\dfrac{x^2-7xy+12y^2}{x^2+5xy+6y^2} \div \dfrac{x^2-5xy+4y^2}{x^2+xy-2y^2}.$

6. $\left\{\dfrac{a+2b}{b-2a}-\dfrac{b+2a}{a-2b}\right\}\times\left\{\dfrac{3}{2b-a}-\dfrac{1}{b-a}\right\}.$

7. $\dfrac{x^6+x^5-x^4-x}{x^6-1}.$

8. $\dfrac{2}{x^2+x}+\dfrac{2x-1}{x^2-x+1}-\dfrac{2x^3-1}{x^4+x}.$

9. $\dfrac{a^4-b^4}{a^2+ab+b^2}\times\dfrac{a^3-b^3}{(a-b)(a-2b)}\times\dfrac{a^2-4b^2}{(a+b)(a+2b)}\times\dfrac{1}{(a-b)(a^2+b^2)}.$

10. $\dfrac{1}{x-3}-\dfrac{1}{x+4}+\dfrac{1+\dfrac{x}{x+1}}{x-\dfrac{12}{x+1}}.$

11. $\dfrac{2x^3-17x^2+29x-12}{4x^3-36x^2+27x+27}.$

12. $\left(1-\dfrac{2a}{\dfrac{z^2}{a}+a}\right)\left(1-\dfrac{3a}{\dfrac{z^2}{a}-a}\right)\div\left(1-\dfrac{a}{z-a}\right)\left(1+\dfrac{a}{z+a}\right).$

13. $\dfrac{3}{2(x-1)}-\dfrac{1}{2(x+1)}+\dfrac{x-2}{x^2+1}-\dfrac{2(x^3+2)}{x^4-1}.$

14. $\dfrac{1-x}{1-x+x^2}-\dfrac{\dfrac{1}{x}\left(\dfrac{1}{x}-2\right)}{\dfrac{1}{x^3}+1}.$

15. $\dfrac{\dfrac{2}{x+1}-\dfrac{1}{x}}{1+\dfrac{x}{1-2x}}+\dfrac{\dfrac{2}{x+1}-1}{x+\dfrac{x}{x-2}}.$

16. $\dfrac{\left(1+\dfrac{1}{x}\right)\times\left(1-\dfrac{1}{x}\right)^2}{x-\dfrac{1}{x}}.$

17. $\dfrac{1+\dfrac{y^2+z^2-x^2}{2yz}}{1-\dfrac{x^2+y^2-z^2}{2xy}}.$

18. $\dfrac{a}{(a-b)(a-c)}-\dfrac{2b}{(b-c)(b-a)}+\dfrac{3c}{(c-a)(c-b)}.$

19. $\dfrac{2x^2-17x+21}{15x^2+16x-15}\div\dfrac{2x^2-11x+12}{3x^2-10x-25}\times\dfrac{5x^2-23x+12}{x^2-2x-35}.$

20. $\dfrac{x^2-(y-z)^2}{(z+x)^2-y^2}+\dfrac{y^2-(z-x)^2}{(x+y)^2-z^2}-\dfrac{(x-y)^2-z^2}{(y+z)^2-x^2}.$

21. $\dfrac{(x+y)^2+(x-y)^2}{(x+y)^2-(x-y)^2}\div\dfrac{x^4-y^4}{2xy(x-y)}.$

22. $\dfrac{x^3+3x^2+5x+15}{x^3+2x^2+5x+10}+\dfrac{x^4+x^3+3x^2+x-2}{x^4+2x^3+3x^2+4x-4}.$

23. $$\frac{x}{1+\dfrac{x}{1-x+\dfrac{x}{1+x}}} \div \frac{1+x+x^2}{1+3x+3x^2+2x^3}.$$

24. $$\left(\frac{\dfrac{x}{y}+2}{\dfrac{x}{y}+1}+\frac{x}{y}\right)+\left(\frac{x}{y}+2-\frac{\dfrac{x}{y}}{\dfrac{x}{y}+1}\right).$$

25. $$\frac{\dfrac{x+y}{x-y}-\dfrac{x^2+y^2}{x^2-y^2}}{\dfrac{x+y}{x-y}+\dfrac{x^2+y^2}{x^2-y^2}} \div \frac{\dfrac{1}{y}-\dfrac{1}{x}}{x^3-y^3}.$$

26. $$\frac{1+x^3}{1-\dfrac{x}{1+\dfrac{x}{1-x}}}-\frac{1-x^3}{1+\dfrac{x}{1-\dfrac{x}{1+x}}}.$$

27. $$a^3+\frac{a^2}{a^2+\dfrac{1}{a^3-\dfrac{a^8+a^3-1}{a^5}}}.$$

28. $$\frac{\dfrac{x^2}{y^2}\left(1+\dfrac{y}{x}\right)^2-\left(\dfrac{x}{y}+1\right)}{\dfrac{x^2}{y^2}\left(1+\dfrac{y}{x}\right)^2-\dfrac{x}{y}\left(\dfrac{x}{y}+1\right)}.$$

29. $$\frac{\dfrac{2}{b+c}-\dfrac{1}{b}}{c+\dfrac{bc}{c-2b}}+\frac{\dfrac{2}{b+c}-\dfrac{1}{c}}{b+\dfrac{bc}{b-2c}}.$$

30. $$\frac{(x-y)^4-xy(x-y)^2-2x^2y^2}{(x-y)(x^3-y^3)+2x^2y^2}.$$

31. $$\frac{x^3-1}{x^3+1}+\frac{x^3+1}{x^3-1}-\frac{2x^4}{x^4+x^2+1}.$$

32. $$\frac{\dfrac{a^3}{b^3}-\dfrac{b^3}{a^3}}{\left(\dfrac{a}{b}-\dfrac{b}{a}\right)\left(\dfrac{a}{b}+\dfrac{b}{a}-1\right)} \div \frac{\dfrac{1}{a^2}+\dfrac{1}{ab}+\dfrac{1}{b^2}}{\dfrac{1}{b}-\dfrac{1}{a}}.$$

33. $$\frac{3}{2c-3}-\frac{2c+15}{4c^2+9}-\frac{2}{2c+3}+\frac{18(2c+15)}{81-16c^4}.$$

34. $$\left(2-\frac{3n}{m}+\frac{9n^2-2m^2}{m^2+2mn}\right) \div \left(\frac{1}{m}-\frac{1}{m-2n-\dfrac{4n^2}{m+n}}\right).$$

35. $$\frac{\dfrac{1-b}{1+b}-\dfrac{1-a}{1+a}}{1+\dfrac{(1-a)(1-b)}{(1+a)(1+b)}}.$$

36. $$\frac{x+2-\dfrac{1}{x+2}}{x+2-\dfrac{4}{x+5}} \times \frac{x+4-\dfrac{4}{x+4}}{x+4-\dfrac{1}{x+4}}.$$

37. $$\frac{(ac+bd)^3-(ad+bc)^3}{(a-b)(c-d)}-\frac{(ac+bd)^3+(ad+bc)^3}{(a+b)(c+d)}.$$

38. $$\frac{y+\dfrac{1}{y}+1}{y+\dfrac{1}{y}}-\frac{y+\dfrac{1}{y}-1}{y-\dfrac{1}{y}}+\frac{1-\dfrac{1}{y}}{1+\dfrac{1}{y}}+\frac{1+\dfrac{1}{y}}{1-\dfrac{1}{y}}-\frac{2y+\dfrac{2}{y^2}}{y^2-\dfrac{1}{y^2}}.$$

TEST PAPERS G.

I.

1. Find the value of

$$\sqrt{(7x+2y+\tfrac{1}{6})}+\sqrt[3]{(5y+7z+2\tfrac{1}{6})}$$

when $4x=5$, $3y=-4$, $2z=9$.

2. Simplify (1) $\dfrac{\dfrac{a}{a+b}+\dfrac{b}{a-b}}{\dfrac{a}{a-b}-\dfrac{b}{a+b}}$; (2) $\dfrac{1}{m+\dfrac{1}{1+\dfrac{m+1}{3-m}}}$.

3. Solve (1) $\dfrac{1}{7}(3x-9)-\dfrac{1}{11}(x+1)=\dfrac{1}{8}(3x-14)$;

(2) $7x+\dfrac{5y+9x}{11}=17$, $9x+\dfrac{11y+9x}{17}=21$.

4. I bought a certain number of articles at 7 for 6*d.*; if they had been 13 for 1*s.* I should have spent 6*d.* more: how many did I buy?

5. *A* pays a debt of £6 in shillings and half-crowns. If he had half as many shillings again, and three times as many half-crowns he could pay his debt twice over. How many shillings and half-crowns has he?

II.

1. Resolve into factors:

(1) $9(x-2)^2-4(x-1)^2$; (2) $2x^3-x^2+8x-4$.

2. Simplify (1) $\left(\dfrac{a}{1+a}+\dfrac{1-a}{a}\right)\div\left(\dfrac{a}{1+a}-\dfrac{1-a}{a}\right)$;

(2) $\dfrac{1}{2a^2(a^2+c^2)}-\dfrac{1}{4a^3(c-a)}+\dfrac{1}{4a^3(a+c)}$.

3. Solve the equations:

(1) $\dfrac{2}{7}(5x+4)-\dfrac{1}{4}(3x-1)=\dfrac{9}{5}$;

(2) $\cdot 2x-\cdot 5y=\cdot 25$, $2\cdot 5x-y=1\cdot 025$.

4. Two gallons more than $\frac{2}{5}$ of the contents of a cistern having leaked away, the remainder is found to be 18 gallons less than $\frac{11}{15}$ of the original quantity. What was the leakage?

5. The sum of the two digits of which a number is composed is 8; if 36 be added to the number, the digits are reversed. Find the number.

III.

1. Find the H.C.F. and L.C.M. of

$$3x^2+x-10, \quad 6x^2-x-15, \quad 3x^3-5x^2-12x+20.$$

2. Simplify

$$(1)\quad \frac{\dfrac{x}{x+1}+\dfrac{x}{x-1}}{\dfrac{2}{x^2-1}}-\frac{4x-\dfrac{1}{x}}{2+\dfrac{1}{x}}; \qquad (2)\quad \frac{1}{a-\dfrac{1}{a+\dfrac{1}{a}}}-\frac{1}{a+\dfrac{1}{a-\dfrac{1}{a}}}.$$

3. Solve (1) $\frac{1}{2}(2x+7)-\frac{4}{5}\left(3x-\frac{5}{2}\right)=9$;

(2) $\frac{5}{x}-3y=1, \quad \frac{3}{x}+5y=21.$

4. *A* had £50 more than *B*; *A* paid a third part of his money to *B*, and *B* paid back a fifth part of what he then had: the result was that *B* had £30 more than *A*. How much had each at first?

5. The second of three numbers is as much greater than the first as the third is greater than the second. If five times the first is added to four times the second the result is 600, and if five times the third is subtracted from eight times the second the remainder is 150. Find the numbers.

IV.

1. Find the product of

$$\frac{1}{3}\{(a^2+1)-2a(1-a)\}+\frac{1}{6} \text{ and } \frac{1}{2}\left\{3(a^2-1)-\frac{a}{3}(3a-4)\right\}+1,$$

and obtain its numerical value when $a=-3$.

2. Simplify (1) $\left\{\frac{p}{p-1}-\frac{p}{p+1}\right\}\div\frac{1}{3}\left\{\frac{1}{p-1}-\frac{1}{p+1}+\frac{1}{1-p^2}\right\}$;

(2) $\frac{3x^3-2x^2-x}{4x^3-2x^2-3x+1}$.

3. Solve (1) $\frac{1}{x-a}-\frac{1}{x-b}=\frac{a-b}{x^2-ab}$;

(2) $\frac{2x+5}{3}=\frac{y+4}{2}=\frac{2x+2y+9}{6}$.

4. A man bought a certain number of sheep for £74. Having lost 7 of them, he sold one-fifth of the rest for 12 guineas, making a profit of 5 per cent. on what he paid for them. How many did he buy?

5. Half *A*'s age exceeds three-eighths of *B*'s age by 7 years, and five-sixths of *B*'s age is equal to five-eighths of *A*'s; find the age of each.

V.

1. Find the value of $\sqrt[3]{\{5(b^2-c^2)-a^2\}}+\sqrt[4]{3\{a(a^2-c^2)-1\}}$, when $a=4$, $b=5$, $c=3$.

2. Simplify

$$(1)\quad \left(1+\frac{x^2-y^2-z^2}{2yz}\right)\times\left(\frac{1}{x+y-z}+\frac{1}{z+x-y}\right);$$

$$(2)\quad \frac{x^3-3x^2+3x-2}{x^4+x^2+1}.$$

3. Solve the equations:

$$(1)\quad \frac{3-4x}{2x+1}-\frac{24x-5}{2-3x}=6+\frac{10}{2x+1};\qquad (2)\quad \frac{x}{a}+\frac{y}{b}=\frac{x}{b}-\frac{y}{a}=\frac{a}{b}.$$

4. By buying oranges at 20 for a shilling and selling them at 6 for 5*d.* a hawker gained 4*s.* 8*d.* How many did he buy?

5. I have only half-crowns and sixpences in my purse. Four times the number of sixpences exceeds three times the number of half-crowns by 1, and the half-crowns are worth £2. 13*s.* more than the sixpences. How many coins of each kind have I?

VI.

1. Prove that

$$(a-b)^2+(b-c)^2+(c-a)^2$$
$$=2(a-b)(a-c)+2(b-c)(b-a)+2(c-a)(c-b).$$

2. Simplify

$$(1)\quad \frac{x^2+y^2-z^2+2xy}{2x^3-2xy^2+2xz^2+4x^2z}+\frac{1}{2x};$$

$$(2)\quad \frac{1}{a^2+5a+6}+\frac{2a}{a^2+4a+3}+\frac{1}{a^2+3a+2}+\frac{2a+4}{a+3}.$$

3. Solve the equations:

$$(1)\quad \frac{5x-8}{x-3}+3-\frac{16x-17}{2x-9}=\frac{56}{6-2x};$$

$$(2)\quad \frac{2x-5}{3}-\frac{x-y}{2}=4,\quad \frac{y}{2}-\frac{2x-y}{5}=3\tfrac{1}{2}.$$

4. Of the candidates in a certain examination 36 per cent. failed. If there had been 7 more candidates, of whom one passed, the failures would have been 37·5 per cent.: how many candidates were there?

5. A farmer buys 20 sheep and 15 cows for £330. Had he bought half as many sheep again but at £1 a head less, and 3 fewer cows at 75 per cent. of the price he paid, he would have spent £81 less. What did he pay for each sheep and each cow?

CHAPTER XXII.

INVOLUTION.

185. DEFINITION. **Involution** is the general name for multiplying an expression by itself so as to find its second, third, fourth, or any other power.

Involution may always be effected by actual multiplication. Here, however, we shall deal with the easiest of those cases in which the results may be written down at once.

By definition we have $a^3 = a \,.\, a \,.\, a$,

also $(-a)^3 = (-a)(-a)(-a)$;

and, by the rules of multiplication,

$$a \,.\, a \,.\, a = a^{1+1+1}$$

$$(-a)(-a)(-a) = -a^{1+1+1}.$$

Similarly

$$(a^2)^3 = a^2 \,.\, a^2 \,.\, a^2 = a^{2+2+2} = a^{2\times3} = a^6,$$

$$(-a^2)^3 = (-a^2)(-a^2)(-a^2) = -a^{2+2+2} = -a^{2\times3} = -a^6,$$

$$(-x^3)^2 = (-x^3)(-x^3) = x^{3+3} = x^{3\times2} = x^6,$$

$$(-x^5)^3 = (-x^5)(-x^5)(-x^5) = -x^{5+5+5} = -x^{5\times3} = -x^{15},$$

$$(4m^4)^2 = (4)^2(m^4)^2 = 16m^{4\times2} = 16m^8.$$

Hence we obtain a rule for raising a single expression to any required power.

Rule. *Raise each* LITERAL *factor of the expression to the required power by* MULTIPLYING *its index by the index of that power.*

If there is a numerical coefficient, raise it to the required power by Arithmetic, and prefix the result, with its proper sign, to the literal expression already obtained.

186. The following general principles, which are evident from the Rule of Signs, should here again be carefully noted.

(1) The SQUARE *of every expression, whether such expression is positive or negative, is* POSITIVE.

(2) *No* EVEN *power of* ANY *expression can be* NEGATIVE.

(3) *Any* ODD *power of an expression will have the* SAME SIGN *as the expression itself.*

Example 1. $(-2x^2)^5=(-2)^5x^{2\times 5}=-32x^{10}$.

Example 2. $(-3ab^3)^6=(-3)^6a^6b^{3\times 6}=729a^6b^{18}$.

Example 3. $\left(\frac{2ab^3}{3x^2y}\right)^4=\frac{16a^4b^{12}}{81x^8y^4}$.

Note. In Example 3 it will be seen that the numerator and denominator are operated upon separately.

Example 4. $\frac{(-3xy^3)^4}{(6xy^2)^3}=\frac{(-3)^4x^4y^{12}}{(6)^3x^3y^6}=\frac{81xy^6}{216}=\frac{3}{8}xy^6$.

Note. In cases like Example 4, where the numerator and denominator are to be raised to different powers, no cancelling must be performed until after the several factors of both numerator and denominator are raised to the required powers.

EXAMPLES XXII. a.

Write down the square of each of the following expressions :

1. $2a^3$.
2. $3xy^3$.
3. b^2c^3.
4. $4ab^2$.
5. $7c^3d^5$.
6. $6a^3b^6$.
7. $5a^2b^5c$.
8. $-3ab^3c^5$.
9. $-9p^4q^6$.
10. $-4ab^8$.
11. $-a^2b^3c^5d^4$.
12. $-8m^3n^6$.
13. $-\frac{3xy}{4}$.
14. $\frac{2m^2n^3}{3p^5q^2}$.
15. $-\frac{1}{3x^3}$.
16. $-\frac{1}{4y^8}$.
17. $-\frac{7k^3l^5}{8pq^4}$.
18. $-\frac{1}{9a^5b^3c^4}$.
19. $\frac{4x^4y^3z^5}{7}$.
20. $-\frac{11}{10b^5c^8}$.

Write down the cube of each of the following expressions :

21. x^3y^2.
22. $2x^2$.
23. $3y^3$.
24. $6x^2z^3$.
25. $-3a^9$.
26. $-2x^2y^3z$.
27. $-4p^4q^2$.
28. $-5c^5d^6$.
29. $\frac{1}{3a^3b}$.
30. $-\frac{2k^2l^3}{p^5q}$.
31. $-\frac{3x^4y^2}{7}$.
32. $-\frac{6}{5a^5b^2}$.

Write down the value of each of the following expressions :

33. $(x^2y)^3$.
34. $(-xy^3)^4$.
35. $(-2m^3n^2)^5$.
36. $(-3p^2q^3)^3$.
37. $\left(\frac{1}{4a^5}\right)^3$.
38. $\left(-\frac{2a^3x}{p^2q^5}\right)^5$.
39. $\left(-\frac{1}{a^2b^4c^5}\right)^8$.
40. $\left(\frac{a^4b^5c}{x^3y^6z^2}\right)^9$.
41. $\frac{(a^2b)^3}{(ab^2)^4}$.
42. $\frac{(-m^4n^3)^2}{(m^3n^2p)^4}$.
43. $\frac{(-3x^2y)^3}{(-2x^3yz^2)^5}$.
44. $\frac{(4c^3)^2}{(-3c^2d^7)^3}$.

187. In Art. 72 the following rules were stated:

Rule 1. *The square of the* SUM *of two quantities is equal to the sum of their squares* INCREASED *by twice their product.*

Rule 2. *The square of the* DIFFERENCE *of two quantities is equal to the sum of their squares* DIMINISHED *by twice their product.*

Thus
$$(x+2y)^2=x^2+2\,.\,x\,.\,2y+(2y)^2$$
$$=x^2+4xy+4y^2.$$
$$(2a^3-3b^2)^2=(2a^3)^2-2\,.\,2a^3\,.\,3b^2+(3b^2)^2$$
$$=4a^6-12a^3b^2+9b^4.$$

By an application of the above we may obtain a rule for writing down the square of an expression which consists of more than two terms.

Thus since
$$(a+b)^2=a^2+2ab+b^2,$$
$$(x+y+z)^2=\{(x+y)+z\}^2=(x+y)^2+2(x+y)z+z^2$$
$$=x^2+2xy+y^2+2xz+2yz+z^2$$
$$=x^2+y^2+z^2+2xy+2xz+2yz.$$

Also since
$$(a-b)^2=a^2-2ab+b^2,$$
$$(x+y-z)^2=\{(x+y)-z\}^2=(x+y)^2-2(x+y)z+z^2$$
$$=x^2+y^2+z^2+2xy-2xz-2yz;$$
$$(x-y-z)^2=\{(x-y)-z\}^2=(x-y)^2-2(x-y)z+z^2$$
$$=x^2+y^2+z^2-2xy-2xz+2yz.$$

In each of these instances we observe that the square consists of

(1) the sum of the squares of the several terms of the expression, together with

(2) twice the sum of the products of those terms taken two and two with their proper signs; that is, each product has the sign + or − *according as the two factors composing it have like or unlike signs.*

Note. The *square terms* are always positive.

The same laws hold whatever be the number of terms in the expression to be squared, and therefore the following general Rule may be stated:

Rule. *The square of any multinomial is equal to the sum of the squares of the several terms, together with the algebraical sum of twice the product of each term into each of the terms which follow it.*

Ex. 1. $(x-2y-3z)^2=x^2+4y^2+9z^2-2\,.\,x\,.\,2y-2\,.\,x\,.\,3z+2\,.\,2y\,.\,3z$
$$=x^2+4y^2+9z^2-4xy-6xz+12yz.$$

Ex. 2. The square of (x^3-x^2+x-1)

$$=x^6+x^4+x^2+1+2x^3(-x^2+x-1)-2x^2(x-1)+2x(-1)$$
$$=x^6+x^4+x^2+1-2x^5+2x^4-2x^3-2x^3+2x^2-2x$$
$$=x^6-2x^5+3x^4-4x^3+3x^2-2x+1,$$

by collecting like terms and rearranging.

EXAMPLES XXII. b.

Write down the square of each of the following expressions:

1. $a+2b$.
2. $2a-b$.
3. $x+3y$.
4. $2x-3y$.
5. $m+4n$.
6. $p-5q$.
7. $4-x$.
8. $y+3$.
9. $a+7$.
10. $5-c$.
11. $cd+1$.
12. $2ab-3$.
13. $1+x^2$.
14. $1+3xy$.
15. x^2-2x.
16. $2c-cd$.
17. $a+b-c$.
18. $a-b-c$.
19. $x+y+2z$.
20. $x-2y+z$.
21. $2p-q-r$.
22. x^2-x-1.
23. $2x^2-x+1$.
24. $k^2-l^2+m^2$.
25. $3k^2-5k+2$.
26. $a-b+c+d$.
27. $2x+y-3a+b$.
28. $m+n-p+2q$.
29. $a-\frac{1}{2}b+\frac{c}{4}$.
30. $\frac{x}{3}-3y-\frac{3}{2}$.
31. $\frac{3}{2}-m+\frac{2}{3}m^2$.

188. By actual multiplication, we have

$$(a+b)^3=(a+b)(a+b)(a+b)$$
$$=a^3+3a^2b+3ab^2+b^3.$$

Also $$(a-b)^3=a^3-3a^2b+3ab^2-b^3.$$

By observing the law of formation of the terms in these results we can write down the cube of any binomial.

Example 1. $$(2x+y)^3=(2x)^3+3(2x)^2y+3(2x)y^2+y^3$$
$$=8x^3+12x^2y+6xy^2+y^3.$$

Example 2. $$(3x-2a^2)^3=(3x)^3-3(3x)^2(2a^2)+3(3x)(2a^2)^2-(2a^2)^3$$
$$=27x^3-54x^2a^2+36xa^4-8a^6.$$

EXAMPLES XXII. c.

Write down the cube of each of the following expressions:

1. $p+q$.
2. $m-n$.
3. $a-2b$.
4. $2c+d$.
5. $3x+2y$.
6. $4x-1$.
7. $1-5y$.
8. $2ab-3$.
9. a^2+3b^2.
10. p^2-3q^2.
11. $3c^2-2d^3$.
12. $4x^3-3x$.
13. $a-\frac{b}{2}$.
14. $\frac{c}{3}+1$.
15. $\frac{l^2}{3}-3l$.
16. $\frac{x}{6}+2y$.

CHAPTER XXIII.

EVOLUTION.

189. DEFINITION. THE **root** of any proposed expression is that quantity which will produce the given expression by being multiplied by itself the number of times denoted by the index of the root.

The operation of finding the root is called **Evolution**: it is the reverse of Involution.

Thus $\sqrt[3]{x^6}=x^2$, because $(x^2)^3=x^6$.

$\sqrt[3]{-x^6}=-x^2$, because $(-x^2)^3=-x^6$.

190. It should here again be carefully observed that every positive quantity has two square roots equal in magnitude but opposite in sign (Art. 69).

Example 1. $\sqrt{9a^2x^6}=+3ax^3$, or $-3ax^3$.

[The two roots are conveniently written $\pm 3ax^3$, and ± is read "*plus or minus.*"]

Example 2. $\sqrt{169m^{10}n^8}=\pm 13m^5n^4$.

191. Again, by the Rule of Signs, it is evident that

(1) any *even* root of a *positive* quantity may be either *positive* or *negative*;

(2) *no negative* quantity can have an *even* root;

(3) every *odd* root of a quantity has the same sign as the quantity itself.

Examples. $\sqrt[4]{a^{12}b^8}=\pm a^3b^2$, because $(a^3b^2)^4$ or $(-a^3b^2)^4=a^{12}b^8$.

$\sqrt[3]{-x^9}=-x^3$, because $(-x^3)^3=-x^9$.

$\sqrt[5]{c^{20}}=c^4$, because $(c^4)^5=c^{20}$.

$\sqrt[4]{81x^{20}}=\pm 3x^5$, because $(3x^5)^4$ or $(-3x^5)^4=81x^{20}$.

Throughout the present chapter, however, in dealing with the even root of a positive quantity, we shall confine our attention to the *positive* value only.

192. From the foregoing examples we may deduce a general rule for extracting any required root of a simple expression.

Rule. *Find the index of each literal factor by* DIVIDING *its index in the given expression by the index of the root required.*

If there is a numerical coefficient find its root by Arithmetic, and prefix it, with its proper sign, to the literal expression already obtained.

Examples.

$$\sqrt[3]{-64x^6} = -4x^{\frac{6}{3}} = -4x^2.$$

$$\sqrt[6]{a^{24}c^{18}} = a^{\frac{24}{6}}c^{\frac{18}{6}} = a^4c^3.$$

$$\sqrt{\frac{49x^{10}}{25c^4d^6}} = \frac{7x^{\frac{10}{2}}}{5c^{\frac{4}{2}}d^{\frac{6}{2}}} = \frac{7x^5}{5c^2d^3}.$$

EXAMPLES XXIII. a.

Write down the square root of each of the following expressions:

1. x^2y^6. 2. $4c^4d^8$. 3. $16a^4b^{16}$. 4. $9x^6y^{12}$.

5. $64p^2q^{16}$. 6. $81x^{10}$. 7. $144x^{24}y^6$. 8. $36m^{36}$.

9. $\frac{1}{81a^{18}}$. 10. $\frac{64x^{18}}{25}$. 11. $\frac{16}{m^{16}n^8}$. 12. $\frac{169y^{26}}{49}$.

13. $\frac{289}{324p^{12}}$. 14. $\frac{100}{81x^{10}y^{18}}$. 15. $\frac{36c^{12}}{25a^{10}}$. 16. $\frac{324a^{24}}{400b^{40}c^{20}}$.

Write down the cube root of each of the following expressions:

17. $8a^3b^6$. 18. $27c^6d^9$. 19. $-64x^3y^9$. 20. $343a^{18}$.

21. $-\frac{125}{b^9c^{12}}$. 22. $-\frac{27a^{27}}{8b^9}$. 23. $\frac{m^{15}}{8p^9q^{12}}$. 24. $-\frac{1}{729a^{27}}$.

Write down the value of each of the following expressions:

25. $\sqrt[4]{a^8b^{12}}$. 26. $\sqrt[5]{x^{15}y^{20}}$. 27. $\sqrt[6]{64a^{18}b^{12}}$.

28. $\sqrt[5]{-32d^{10}}$. 29. $\sqrt[7]{128a^{63}}$. 30. $\sqrt[9]{-m^{18}n^{27}p^{36}}$.

31. $\sqrt[5]{-\frac{243}{x^{15}}}$. 32. $\sqrt[7]{-\frac{a^7b^{14}}{c^{28}}}$. 33. $\sqrt[11]{-\frac{1}{x^{33}y^{55}}}$.

34. $\sqrt[12]{\frac{1}{k^{24}l^{72}}}$. 35. $\sqrt[13]{-\frac{a^{39}}{b^{52}c^{65}}}$. 36. $\sqrt[5]{-\frac{32k^{20}l^{15}}{243n^{50}}}$.

193. By the Rules recapitulated in Art. 187, we are able to write down the square of any binomial.

Thus $(2x+3y)^2 = (2x)^2 + 2 \cdot 2x \cdot 3y + (3y)^2$.

Conversely, by observing the form of the terms of an expression, it may sometimes be recognised as a complete square, and its square root written down at once.

Example 1. Find the square root of $25x^2-40xy+16y^2$.

$$\begin{aligned}\text{The expression}&=(5x)^2-2\,.\,20xy+(4y)^2\\&=(5x)^2-2(5x)(4y)+(4y)^2\\&=(5x-4y)^2.\end{aligned}$$

Thus the required square root is $5x-4y$.

Example 2. Find the square root of $\dfrac{64a^2}{9b^2}+4+\dfrac{32a}{3b}$.

$$\begin{aligned}\text{The expression}&=\left(\frac{8a}{3b}\right)^2+(2)^2+2\left(\frac{16a}{3b}\right)\\&=\left(\frac{8a}{3b}\right)^2+2\left(\frac{8a}{3b}\right)(2)+(2)^2\\&=\left(\frac{8a}{3b}+2\right)^2.\end{aligned}$$

Thus the required square root is $\dfrac{8a}{3b}+2$.

Example 3. Find the square root of $4a^2+b^2+c^2+4ab-4ac-2bc$.

Arrange the terms in descending powers of a, and the other letters alphabetically; then

$$\begin{aligned}\text{the expression}&=4a^2+4ab-4ac+b^2-2bc+c^2\\&=4a^2+4a(b-c)+(b-c)^2\\&=(2a)^2+2\,.\,2a(b-c)+(b-c)^2\\&=\{2a+(b-c)\}^2;\end{aligned}$$

whence the required square root is $2a+b-c$.

Or we might proceed as follows:

$$\text{the expression}=(2a)^2+b^2+c^2+2\,.\,(2a)b-2\,.\,(2a)c-2\,.\,b\,.\,c,$$

which is evidently the square root of $2a+b-c$. [Art. 186.]

EXAMPLES XXIII. b.

By inspection determine the square root of each of the following expressions:

1. $x^2+10x+25$. 2. $y^2-18y+81$. 3. $121-22m^2+m^4$.
4. $9-12x^3+4x^6$. 5. $9c^2+42c+49$. 6. $16-40y^6+25y^{12}$.
7. $a^4-6a^2b^2+9b^4$. 8. $x^6-24x^3yz+144y^2z^2$.
9. $m^4n^4+26m^2n^2p^3+169p^6$. 10. $49a^2b^6-112ab^3d^5+64d^{10}$.
11. $\dfrac{1}{4}a^2-3ab+9b^2$. 12. $\dfrac{m^2}{9}+\dfrac{mn^2}{3}+\dfrac{n^4}{4}$.
13. $\dfrac{9a^2}{b^2}-\dfrac{30ac}{bd}+\dfrac{25c^2}{d^2}$. 14. $\dfrac{9a^2}{16x^8}+\dfrac{4x^8}{9a^2}-1$.
15 $(x-1)^2+4a^2+4a(x-1)$. 16. $100c^{12}-20c^6(a+b)+(a+b)^2$.
17. $x^2+2xy-4xz+y^2-4yz+4z^2$. 18. $a^2+4b^2+c^2-4ab-2ac+4bc$.

194. When the square root cannot be easily determined by inspection we must have recourse to the rule explained in the next article, which is quite general, and applicable to all cases. *But the pupil is advised to use methods of inspection wherever possible, in preference to rules.*

195. Since the square of $a+b$ is $a^2+2ab+b^2$, we have to discover a process by which a and b, the terms of the root, can be found when $a^2+2ab+b^2$ is given.

Now $$a^2+2ab+b^2=a^2+b(2a+b),$$
so that the expression is made up of

(1) the *square* of the *first* term of the root, together with

(2) the *product* of the *second* term of the root into an expression consisting of *this second term added to twice the first term of the root.*

By reversing the process we arrive at the following method of working:

$$\begin{array}{r|l}
 & a^2+2ab+b^2\ (\ a+b \\
 & a^2 \\
\hline
2a+b & 2ab+b^2 \\
 & 2ab+b^2 \\
\cline{2-2}
\end{array}$$

Explanation. (1) The terms are first arranged according to the powers of one letter a.

(2) The square root of a^2 is written down as *the first term of the root*, and its square subtracted from the given expression.

(3) The first term of the remainder is *divided by twice the first term of the root* to obtain the second term of the root, that is, b.

(4) *The second term of the root is added to twice the term already found* to form the complete divisor $2a+b$.

Example. Find the square root of $9x^2-42xy-49y^2$.

$$\begin{array}{r|l}
 & 9x^2-42xy+49y^2\ (\ 3x-7y \\
 & 9x^2 \\
\hline
6x-7y & -42xy+49y^2 \\
 & -42xy+49y^2 \\
\cline{2-2}
\end{array}$$

Explanation. The square root of $9x^2$ is $3x$, and this is the first term of the root.

By doubling this we obtain $6x$, which is the first term of the divisor. Dividing $-42xy$, the first term of the remainder, by $6x$ we get $-7y$, the new term in the root, which has to be annexed both to the root and divisor. We next multiply the complete divisor by $-7y$ and subtract the result from the first remainder. There is now no remainder, and the root has been found.

196. The rule can be extended so as to find the square root of any multinomial. The first two terms of the root will be obtained as before. When we have brought down *the second remainder*, the first part of the new divisor is obtained by doubling the terms of the root already found. We then divide the first term of the remainder by the first term of the new divisor, and set down the result as the next term in the root and in the divisor. We next multiply the complete divisor by the last term of the root and subtract the product from the last remainder. If there is now no remainder the root has been found; if there is a remainder we continue the process.

Example. Find the square root of

$$25x^2a^2 - 12xa^3 + 16x^4 + 4a^4 - 24x^3a.$$

Rearrange in descending powers of x.

$$\begin{array}{l|l}
 & 16x^4 - 24x^3a + 25x^2a^2 - 12xa^3 + 4a^4 \;(\; 4x^2 - 3xa + 2a^2 \\
 & 16x^4 \\ \hline
8x^2 - 3xa & -24x^3a + 25x^2a^2 \\
 & -24x^3a + 9x^2a^2 \\ \hline
8x^2 - 6xa + 2a^2 & 16x^2a^2 - 12xa^3 + 4a^4 \\
 & 16x^2a^2 - 12xa^3 + 4a^4 \\ \hline
\end{array}$$

Explanation. When we have obtained two terms in the root, $4x^2 - 3xa$, we have a remainder

$$16x^2a^2 - 12xa^3 + 4a^4.$$

Doubling the terms of the root already found, we place the result, $8x^2 - 6xa$, as the first part of the divisor. Dividing $16x^2a^2$, the first term of the remainder, by $8x^2$, the first term of the divisor, we get $+2a^2$, which we annex both to the root and divisor. We now multiply the complete divisor by $2a^2$ and subtract. There is no remainder, and the root is found.

EXAMPLES XXIII. c.

Find the square root of each of the following expressions:

1. $a^4 - 4a^3 + 6a^2 - 4a + 1$.
2. $a^4 + 2a^3 + 5a^2 + 4a + 4$.
3. $x^4 - 2x^3 - x^2 + 2x + 1$.
4. $4x^4 - 4x^3 + 5x^2 - 2x + 1$.
5. $x^4 - 6x^3 + 13x^2 - 12x + 4$.
6. $4x^4 - 12x^3 + 25x^2 - 24x + 16$.
7. $8p^3 + 1 + 4p^4 - 4p$.
8. $a^4 + 13a^2 + 4 - 12a - 6a^3$.
9. $a^2 - 2ax + x^2 + 2a - 2x + 1$.
10. $x^4 - 2ax^3 + 5a^2x^2 - 4a^3x + 4a^4$.
11. $4c^4 + 6cd^3 + 12c^3d + d^4 + 13c^2d^2$.
12. $x^6 + y^6 + x^2y^4 - 2xy^5 + 2x^3y^3 - 2x^4y^2$.
13. $a^6 + 4a^2b^3 - 2a^3b^2 - 4a^5b + b^4 + 4a^4b^2$.

Find the square root of each of the following expressions :

14. $14a^3 - 4a^4 + 4a^2 + a^6 + 49 - 28a$.

15. $16m^6 - 4m^9 + m^{10} + 16m^7 - 4m^8$.

16. $10b^2 - 20b^3 + 1 - 4b - 24b^5 + 25b^4 + 16b^6$.

17. $25x^2 - 20x^3 + 16 + x^6 + 10x^4 - 4x^5 - 24x$.

18. $9 - 8x^5 - 22x^3 - 12x + x^6 + 20x^4 + 28x^2$.

19. $4a^6 + 13a^4 - 12a^5 + 16 - 22a^3 - 8a + 25a^2$.

20. $a^6 - 7a^2x^4 + 4x^6 + 10a^4x^2 - 12ax^5 + 28a^3x^3 - 8a^5x$.

197. When an expression contains powers of a certain letter and also powers of its reciprocal, there is an important point to be observed. Thus in the expression

$$2x + \frac{1}{x^2} + 4 + x^3 + \frac{5}{x} + 7x^2 + \frac{8}{x^3},$$

the order of *descending* powers of x is

$$x^3 + 7x^2 + 2x + 4 + \frac{5}{x} + \frac{1}{x^2} + \frac{8}{x^3};$$

and the numerical quantity 4 stands between x and $\frac{1}{x}$.

For explanation of this arrangement see Hall and Knight's *Elementary Algebra*, Chap. XXX.

Example 1. Find the square root of $\frac{x^2}{9} + 1\frac{2}{3} - \frac{2}{x} + \frac{1}{x^2} - \frac{2x}{3}$.

First arrange the expression in descending powers of x.

$$\begin{array}{l|l}
 & \frac{x^2}{9} - \frac{2x}{3} + \frac{5}{3} - \frac{2}{x} + \frac{1}{x^2} \left(\frac{x}{3} - 1 + \frac{1}{x} \right. \\
 & \frac{x^2}{9} \\
\hline
\frac{2x}{3} - 1 & -\frac{2x}{3} + \frac{5}{3} \\
 & -\frac{2x}{3} + 1 \\
\hline
\frac{2x}{3} - 2 + \frac{1}{x} & \frac{2}{3} - \frac{2}{x} + \frac{1}{x^2} \\
 & \frac{2}{3} - \frac{2}{x} + \frac{1}{x^2} \\
\hline
\end{array}$$

The last term of the root, $\frac{1}{x}$, is obtained by dividing $\frac{2}{3}$, the first term of the remainder, by $\frac{2x}{3}$, the first term of the divisor; thus

$$\frac{2}{3} \div \frac{2x}{3} = \frac{2}{3} \times \frac{3}{2x} = \frac{1}{x}.$$

Example 2. Find the square root of $24+\frac{16y^2}{x^2}-\frac{8x}{y}+\frac{x^2}{y^2}-\frac{32y}{x}$.

First arrange the terms in descending powers of y.

$$
\begin{array}{l|l}
 & \frac{16y^2}{x^2}-\frac{32y}{x}+24-\frac{8x}{y}+\frac{x^2}{y^2}\ \Big(\ \frac{4y}{x}-4+\frac{x}{y} \\
 & \frac{16y^2}{x^2} \\ \hline
\frac{8y}{x}-4 & -\frac{32y}{x}+24 \\
 & -\frac{32y}{x}+16 \\ \hline
\frac{8y}{x}-8+\frac{x}{y} & 8-\frac{8x}{y}+\frac{x^2}{y^2} \\
 & 8-\frac{8x}{y}+\frac{x^2}{y^2} \\ \hline
\end{array}
$$

Here the third term of the root, $\frac{x}{y}$, arises from the division of 8 by $\frac{8y}{x}$; thus $8\div\frac{8y}{x}=8\times\frac{x}{8y}=\frac{x}{y}$.

EXAMPLES XXIII. d.

Find the square root of each of the following expressions :

1. $\frac{a^2}{4}-4a+16$.
2. $81+9x^2+\frac{x^4}{4}$.
3. $\frac{x^6}{9}+\frac{2x^3y^3}{3}+y^6$.
4. $\frac{x^4y^2}{25}-\frac{x^3y^3}{10}+\frac{x^2y^4}{16}$.
5. $1+2a-a^3+\frac{a^4}{4}$.
6. $a^4-2a^3+\frac{3}{2}a^2-\frac{1}{2}a+\frac{1}{16}$.
7. $\frac{x^4}{64}+\frac{x^3}{8}-x+1$.
8. $x^4-3x^3+\frac{11}{12}x^2+2x+\frac{4}{9}$.
9. $9x^4-2x^3y+\frac{163}{9}x^2y^2-2xy^3+9y^4$.
10. $2x^3y^3+1-\frac{15}{4}x^2y^2-xy+4x^4y^4$.
11. $x^4+\frac{1}{x^4}+3-\frac{2}{x^2}-2x^2$.
12. $\frac{25}{x^2}+4x^4+\frac{16}{x^8}-12x-\frac{24}{x^5}$.
13. $\frac{x}{a}-1+\frac{4a^2}{x^2}+\frac{x^2}{4a^2}-\frac{4a}{x}$.
14. $3-\frac{6x}{a}-\frac{2a}{3x}+\frac{a^2}{9x^2}+\frac{9x^2}{a^2}$.

CHAPTER XXIV.

QUADRATIC EQUATIONS.

198. DEFINITION. An equation which contains the square of the unknown quantity, *but no higher power*, is called a **quadratic equation,** or an **equation of the second degree.**

If the equation contains only the square of the unknown it is called a **pure** quadratic; if it contains both the square and the first power of the unknown it is said to be an **adfected** quadratic.

Thus $5x^2=20$ is a *pure* quadratic,

and $2x^2-5x=3$ is an *adfected* quadratic.

Pure Quadratics.

199. A pure quadratic may be considered as a simple equation in which the *square* of the unknown quantity is to be found.

Example. Solve $\dfrac{9}{x^2-27}=\dfrac{25}{x^2-11}$.

Multiplying up,

$$9x^2-99=25x^2-675;$$
$$\therefore \quad 16x^2=576;$$
$$\therefore \quad x^2=36;$$

and taking the square root of these equals, we have

$$x=\pm 6.$$

Note. We prefix the double sign to the number on the right-hand side for the reason given in Art. 190.

200. In extracting the square root of the two sides of the equation $x^2=36$, it might seem that we ought to prefix the double sign to the quantities on both sides, and write $\pm x=\pm 6$. But an examination of the various cases shews this to be unnecessary. For $\pm x=\pm 6$ gives the four cases:

$$+x=+6, \quad +x=-6, \quad -x=+6, \quad -x=-6,$$

and these are all included in the two already given, namely $x=+6$, $x=-6$. Hence, when we extract the square root of the two sides of an equation, it is sufficient to put the double sign before the square root of *one* side.

Example 1. Solve $3x(x+3)-44=(2x-5)(x+7)$.

Simplifying, we have $3x^2+9x-44=2x^2+9x-35$;

$$\therefore\quad x^2=9;$$

$$\therefore\quad x=\pm 3.$$

Example 2. Solve $\dfrac{9x}{7}+\dfrac{x-7}{x}=1$.

Clearing of fractions, $9x^2+7x-49=7x$;

$$\therefore\quad 9x^2=49;$$

$$\therefore\quad x^2=\frac{49}{9};$$

$$\therefore\quad x=\pm\frac{7}{3}.$$

EXAMPLES XXIV. a.

Solve the equations :

1. $(x+2)(x-2)=12$.

2. $(x+3)(x-3)=16$.

3. $(9+x)(9-x)=45$.

4. $4(x^2+1)=5(x^2-1)$.

5. $2x^2-3=3x^2-7$.

6. $3(x^2-11)=2(x^2+24)$.

7. $2(x^2-4)=3(2x^2-11)$.

8. $15-5(2x^2-3)=3(2x^2+7)$.

9. $\dfrac{x^2-11}{x^2+14}=\dfrac{1}{2}$.

10. $\dfrac{x^2+1}{2x^2-23}=\dfrac{2}{3}$.

11. $\dfrac{15x^2+9}{43x^2-3}=\dfrac{3}{5}$.

12. $\dfrac{x^2+19}{5}=3+\dfrac{x^2}{4}$.

13. $\dfrac{x^2-8}{7}+\dfrac{x^2-3}{3}+\dfrac{5}{21}=0$.

14. $\dfrac{7x^2}{5}-\dfrac{1}{14}(x^2-11)=\dfrac{3}{7}(x^2-25)+34$.

15. $3(x+4)+\dfrac{2x-35}{x-4}=2$.

16. $\dfrac{9x^2-16}{x^2-4}=\dfrac{5x}{x+2}+\dfrac{10}{x-2}$.

17. $\dfrac{x+2}{x+3}+\dfrac{x-2}{x-3}=\dfrac{4}{5}$.

18. $\dfrac{x}{6}+\dfrac{x-3\frac{2}{3}}{x+3}=1\frac{1}{2}$.

Adfected Quadratics.

201. Solution by Completing the Square. The equation $x^2=36$ is an instance of the simplest form of quadratic equations. The equation $(x-3)^2=25$ may be solved in a similar way; for taking the square root of both sides, we have two *simple* equations :

$$x-3=\pm 5.$$

Taking the upper sign, $x-3=+5$, whence $x=8$;

taking the lower sign, $x-3=-5$, whence $x=-2$.

$\therefore$ the solution is $x=8$, or -2.

Now the given equation $(x-3)^2=25$
may be written $x^2-6x+(3)^2=25$,
or $x^2-6x=16$.
Hence, by retracing our steps, we learn that the equation

$$x^2-6x=16.$$

can be solved by first adding $(3)^2$ or 9 to each side, and then extracting the square root; and the reason why we add 9 to each side is that this quantity added to the left side makes it a *perfect square.*

Now whatever the quantity a may be,

$$x^2+2ax+a^2=(x+a)^2,$$

and

$$x^2-2ax+a^2=(x-a)^2;$$

so that if a trinomial is a perfect square, and *its highest power,* x^2 *has unity for its coefficient,* we must always have the term without x equal to *the square of half the coefficient of* x. If, therefore, the terms in x^2 and x are given, the square may be completed by adding the square of half the coefficient of x.

Example. Solve $x^2+14x=32$.

The square of half 14 is $(7)^2$.

$$\therefore\ x^2+14x+(7)^2=32+49;$$

that is,

$$(x+7)^2=81;$$

$$\therefore\ x+7=\pm 9;$$

$$\therefore\ x=-7+9, \text{ or } -7-9;$$

$$\therefore\ x=2, \text{ or } -16.$$

202. When an expression is a perfect square, the *square terms* are always *positive.* **Hence, before completing the square the coefficient of x^2 should be made equal to +1.**

Example. Solve $7x=x^2-8$.

Transposing *so as to have the terms involving* x *on one side, and the square term positive,*

$$x^2-7x=8.$$

Completing the square, $x^2-7x+\left(\frac{7}{2}\right)^2=8+\frac{49}{4}$;

that is,

$$\left(x-\frac{7}{2}\right)^2=\frac{81}{4};$$

$$\therefore\ x-\frac{7}{2}=\pm\frac{9}{2};$$

$$\therefore\ x=\frac{7}{2}\pm\frac{9}{2};$$

$$\therefore\ x=8, \text{ or } -1.$$

EXAMPLES XXIV. b.

Solve the equations :

1. $x^2+x=2.$
2. $x^2+4x=21.$
3. $x^2+8x=33.$
4. $x^2-5x=84.$
5. $x^2+x=110.$
6. $x^2-7x=120.$
7. $x^2-65=8x.$
8. $x+56=x^2.$
9. $x^2-45=12x.$
10. $11x=x^2-26.$
11. $x^2=x+210.$
12. $117=x^2+4x.$
13. $x^2+152=27x.$
14. $7x=60-x^2.$
15. $x^2-51=14x.$
16. $x^2-3x-10=0.$
17. $2+x-x^2=0.$
18. $6+x=x^2.$
19. $380-x^2=x.$
20. $98-7x-x^2=0.$
21. $204=5x+x^2.$
22. $\dfrac{2}{1-x}=\dfrac{3+x}{3}-\dfrac{9+x}{4}.$
23. $\dfrac{3(1+x)}{4}=\dfrac{2x}{3}-\dfrac{2}{1-x}.$
24. $\dfrac{3x+1}{12}+\dfrac{x-3}{x-5}=\dfrac{x+3}{4}+\dfrac{3x-1}{2x-4}-\dfrac{2}{3}.$
25. $\dfrac{4x+1}{12}+\dfrac{x-2}{x-4}=\dfrac{x+1}{3}+\dfrac{3x+2}{2x-2}-\dfrac{1}{4}.$

203. We have shewn that the square may readily be completed when the coefficient of x^2 is unity. All cases may be reduced to this by dividing the equation throughout by the coefficient of x^2.

Example 1. Solve $$32-3x^2=10x.$$

Transposing, $$3x^2+10x=32.$$

Divide throughout by 3, so as to make the coefficient of x^2 unity.

Thus $$x^2+\frac{10}{3}x=\frac{32}{3};$$

completing the square, $$x^2+\frac{10}{3}x+\left(\frac{5}{3}\right)^2=\frac{32}{3}+\frac{25}{9};$$

that is, $$\left(x+\frac{5}{3}\right)^2=\frac{121}{9};$$

$$\therefore\ x+\frac{5}{3}=\pm\frac{11}{3};$$

$$\therefore\ x=-\frac{5}{3}\pm\frac{11}{3}=2,\text{ or }-5\tfrac{1}{3}.$$

Note. We do not add $\left(\frac{10}{6}\right)^2$ but $\left(\frac{5}{3}\right)^2$ to the left-hand side.

Example 2. Solve $5x^2+11x=12$.

Dividing by 5, $x^2+\frac{11}{5}x=\frac{12}{5}$;

completing the square, $x^2+\frac{11}{5}x+\left(\frac{11}{10}\right)^2=\frac{12}{5}+\frac{121}{100}$;

that is, $\left(x+\frac{11}{10}\right)^2=\frac{361}{100}$;

$$\therefore\ x+\frac{11}{10}=\pm\frac{19}{10};$$

$$\therefore\ x=-\frac{11}{10}\pm\frac{19}{10}=\frac{4}{5},\text{ or } -3.$$

204. We see then that the following steps are required for solving an adfected quadratic equation :

(1) *If necessary, simplify the equation so that the terms in* x² *and* x *are on one side of the equation, and the term without* x *on the other.*

(2) *Make the coefficient of* x² **unity** *and* **positive** *by dividing throughout by the coefficient of* x².

(3) *Add to each side of the equation* **the square of half the coefficient of x.**

(4) *Take the square of each side.*

(5) *Solve the resulting simple equations.*

Example. Solve $\frac{3x-8}{x-2}=5-\frac{27}{x+5}$.

Simplifying, $\frac{3x-8}{x-2}=\frac{5x-2}{x+5}$;

multiplying across, $(3x-8)(x+5)=(5x-2)(x-2)$;

that is, $3x^2+7x-40=5x^2-12x+4$,

or $-2x^2+19x=44$.

Dividing by -2, $x^2-\frac{19}{2}x=-22$;

completing the square, $x^2-\frac{19}{2}x+\left(\frac{19}{4}\right)^2=\frac{361}{16}-22$;

that is, $\left(x-\frac{19}{4}\right)^2=\frac{9}{16}$;

$$\therefore\ x-\frac{19}{4}=\pm\frac{3}{4};$$

$$\therefore\ x=\frac{19}{4}\pm\frac{3}{4}=5\tfrac{1}{2},\text{ or } 4.$$

205. In all the instances considered hitherto the quadratic equations have had two roots. Sometimes, however, there is only one solution. Thus if $x^2-2x+1=0$, then $(x-1)^2=0$; whence $x=1$ is the only solution. Nevertheless, in this and similar cases we find it convenient to say that the quadratic has *two equal roots.*

EXAMPLES XXIV. c.

Solve the equations:

1. $2x^2+3x=2$.
2. $3x^2+7x-6=0$.
3. $2x^2-x=15$.
4. $4x^2+11x=3$.
5. $5=3x^2+14x$.
6. $6x^2+35=31x$.
7. $3+11x=4x^2$.
8. $18+5x^2=33x$.
9. $20-9x=20x^2$.
10. $12x^2-17x+6=0$.
11. $6x^2+35x=6$.
12. $28=31x+5x^2$.
13. $27=5x(10x-3)$.
14. $9x(2x-3)=26$.
15. $143=3x(3x-2)$.
16. $7x=3(1-2x^2)$.
17. $4x^2+126=65x$.
18. $2(x^2+20)=21x$.
19. $x=\frac{12}{7}(1-x^2)$.
20. $\frac{7}{3}x-1=\frac{x(x+1)}{2}$.
21. $\frac{2x-1}{3x}=\frac{x-1}{2}$.
22. $(3x-8)(x+5)=(5x-2)(x-2)$.
23. $(x+3)(x-3)-(2x-1)(2x-7)=0$.
24. $10x-\frac{6}{x}+11=0$.
25. $14x-\frac{10}{x}-31=0$.
26. $\frac{3x-5}{3}=\frac{1}{x}+\frac{4x-7}{5}$.
27. $\frac{3x-4}{3}-\frac{4(x-2)}{5}=\frac{1}{x}$.
28. $\frac{3x+4}{19}=\frac{x^2}{4x+3}$.
29. $\frac{3x+2}{x^2}=\frac{11}{2x+3}$.
30. $\frac{x+2}{x-1}-\frac{4-x}{2x}=2\frac{1}{3}$.
31. $\frac{x+3}{x-2}-\frac{1-x}{x}=4\frac{1}{4}$.
32. $\frac{5}{5-x}+\frac{8}{8-x}=3$.
33. $\frac{7}{x+3}-\frac{1}{x-5}=1\frac{2}{3}$.
34. $\frac{x+1}{3x-1}-\frac{x-1}{1+3x}=1$.
35. $\frac{1-x}{1-3x}-\frac{x+1}{3x+1}=\frac{1}{2}$.
36. $\frac{1}{3x-2}+\frac{7}{20}=\frac{3}{5(x-1)}$.
37. $\frac{3}{3-x}+\frac{7}{4-x}=6\frac{1}{2}$.
38. $\frac{x+\frac{1}{6}}{2}-\frac{2}{3x-\frac{1}{2}}=\frac{1}{3}$.
39. $\frac{\frac{2}{3}x-1}{\frac{x}{3}-1}+\frac{3x+8}{2x+1}=5$.
40. $\frac{2x-1}{x+1}+\frac{x-2}{7}+\frac{x-1}{4}=\frac{3(x-2)}{2(x+1)}+\frac{5}{4}(x-1)$.

206. Solution by Factors. The following method of obtaining the solution of a quadratic will sometimes be found more convenient than that of completing the square.

Consider the equation $x^2-x-90=0$.

By resolving the left-hand side into factors we have

$$(x-10)(x+9)=0.$$

Now if *either* of the factors $x-10$, $x+9$ be zero, their product is zero. Hence the quadratic is satisfied by either of the suppositions

$$(1)\ x-10=0, \text{ or } x=10\,; \quad (2)\ x+9=0, \text{ or } x=-9.$$

Thus the roots are 10, -9.

It appears from this that *when a quadratic equation has been simplified, and brought to the form of the equation* $x^2-x-90=0$, its solution can always be readily obtained if the expression on the left-hand side can be resolved into factors. Each of these factors equated to zero gives a simple equation, and a corresponding root of the quadratic.

Example 1. Solve $x^2+\frac{7}{3}x=2$.

Clearing of fractions, $3x^2+7x=6$.

Transposing, *so as to have all the terms on one side of the equation*, we have

$$3x^2+7x-6=0\,;$$
$$\therefore\ (3x-2)(x+3)=0\,;$$

whence $3x-2=0$, or $x+3=0$;

$$\therefore\ x=\frac{2}{3}, \text{ or } -3.$$

Example 2. Solve $2(x^2-6)=3(x-4)$.

We have $2x^2-12=3x-12$;

that is, $2x^2=3x$ ....(1).

Transposing, $2x^2-3x=0$,

$$x(2x-3)=0.$$
$$\therefore\ x=0, \text{ or } 2x-3=0.$$

Thus the roots are 0, $\frac{3}{2}$.

Note. In equation (1) above we might have divided both sides by x and obtained the simple equation $2x=3$, whence $x=\frac{3}{2}$, which is *one* of the solutions of the given equation. But the pupil must be particularly careful to notice that whenever an x, or a factor containing x, is removed by division from every term of an equation it must not be neglected, since this divisor equated to zero will give a root of the equation.

EXAMPLES XXIV. d.

Solve by resolution into factors :

1. $x^2-7x+6=0$. 2. $x^2-3x=28$. 3. $x^2=2x+99$.

4. $x^2-x=132$. 5. $x^2+8x+16=0$. 6. $23x=120+x^2$.

7. $3x^2-10x+3=0$. 8. $6x^2-13x+6=0$. 9. $6x^2+x=2$.

10. $2x^2-5x-12=0$. 11. $x(3x+2)=5$. 12. $2x^2-15=x$.

13. $2x^2-7x=39$. 14. $2x(x+1)=15+x$.

15. $15-11x=8x(1+x)$. 16. $18+5x^2=33x$.

17. $x^2-\frac{3x}{4}+\frac{1}{8}=0$. 18. $9x=\frac{40}{x+1}$. 19. $4x=\frac{15}{x-1}$.

20. $1-\frac{x}{3}(x+1)=x^2+1$. 21. $\frac{2x}{x+1}-\frac{1}{1-x^2}+1=0$.

22. $(3x-4)^2=x^2-2$. 23. $\frac{3x+1}{6}-\frac{3}{3x+2}=\frac{1}{3}$.

24. $\frac{x+10}{x-5}-\frac{10}{x}=\frac{11}{6}$. 25. $\frac{x+2}{x-1}+\frac{x-4}{2x}=\frac{7}{2}$.

26. $\frac{1}{x-2}-\frac{1}{x-3}+\frac{1}{x+1}=0$. 27. $\frac{7}{x+5}-\frac{1}{x-3}=1\frac{2}{3}$.

28. $\frac{12x}{x-2}+\frac{11}{x-1}=\frac{16}{(2-x)(x-1)}$. 29. $\frac{2x}{x-4}+\frac{2x-5}{x-3}=\frac{25}{3}$.

30. $\frac{x-3}{x+3}+6\frac{6}{7}=\frac{x+3}{x-3}$. 31. $\frac{2x}{x-1}+\frac{3x-1}{x+2}=\frac{5x-11}{x-2}$.

32. $\frac{3x-4}{x+1}+\frac{1}{2}=\frac{x-1}{3x+4}$. 33. $\frac{2x+3}{5x-2}-\frac{2}{13}=\frac{3x-2}{4x+1}$.

34. $\dfrac{x+\frac{1}{3}}{2}-\dfrac{1}{x+\frac{2}{3}}=\frac{1}{3}$. 35. $\dfrac{\frac{2}{5}x-1}{x-4}+\dfrac{\frac{3}{5}x+1}{2x-1}=1$.

36. $\frac{7x+10}{10x+3}=\frac{3}{11}+\frac{x-1}{2x-3}$. 37. $\frac{x+1}{2x-5}-\frac{11x+2}{10x-7}=\frac{3}{11}$.

38. $\frac{3x+4}{2x+1}+\frac{x+6}{2}+\frac{x+2}{4}=\frac{x+4}{2x+1}+\frac{10x+9}{5}$.

CHAPTER XXV.

PROBLEMS LEADING TO QUADRATIC EQUATIONS.

207. WE shall now shew how quadratic equations may be applied to the solution of problems.

Example I. Divide 18 into two parts such that the sum of their squares is 170.

Let x be one of the parts, then $18-x$ is the other part.

Thus the symbolical statement of the problem is

$$x^2+(18-x)^2=170,$$

or $$x^2+324-36x+x^2=170;$$

$$\therefore\ 2x^2-36x+154=0;$$

$$\therefore\ x^2-18x+77=0,$$

or $$(x-11)(x-7)=0:$$

$$\therefore\ x=11, \text{ or } 7.$$

Both values of x are admissible, for if $x=11$, then $18-x=7$; and if $x=7$, then $18-x=11$. Thus the two parts are 11 and 7.

Example II. One number exceeds another by 6, and the sum of their reciprocals is $\frac{7}{20}$; what are they?

Let x be the *greater* number, then $x-6$ is the *less*.

Then, from the question, we have

$$\frac{1}{x}+\frac{1}{x-6}=\frac{7}{20},$$

or $$\frac{2x-6}{x(x-6)}=\frac{7}{20};$$

$$\therefore\ 40x-120=7x^2-42x;$$

$$\therefore\ 7x^2-82x+120=0,$$

or $$(x-10)(7x-12)=0;$$

$$\therefore\ x=10, \text{ or } \frac{12}{7},$$

$$\text{and }\ x-6=4, \text{ or } -\frac{30}{7}.$$

Thus the two numbers are 10 and 4, or $\frac{12}{7}$ and $-\frac{30}{7}$, either of which pair of values will be found to satisfy the conditions of the problem.

EXAMPLES XXV a.

1. Find a number which is less than its square by 56.

2. Four times a certain number increased by its square is equal to 21; find it.

3. Divide 13 into two parts so that the sum of their squares is 97.

4. Find two numbers differing by 5, and such that the sum of their squares is 193.

5. Find two numbers differing by 7, and such that the sum of their squares is 137.

6. Find a number whose square, diminished by 14, is equal to seven times the number increased by 4.

7. Find a number whose square, increased by 24, is equal to ten times the number diminished by 1.

8. The sum of the squares of two consecutive numbers is 145; find them.

9. Find a number which, when increased by 7, is equal to 60 times the reciprocal of the number.

10. The sum of the reciprocals of two consecutive numbers is $\frac{17}{72}$; find the numbers.

11. Two numbers differ by 5, and the sum of their reciprocals is $\frac{9}{14}$; find the numbers.

12. One number is three times another number; if each be increased by 1 the sum of the reciprocals is $\frac{10}{21}$. Find the numbers.

Example III. A train travels 300 miles at a uniform rate; if the speed had been 5 miles an hour more, the journey would have taken two hours less: find the rate of the train.

Suppose the train travels at the rate of x miles per hour; then the time occupied is $\dfrac{300}{x}$ hours.

On the other supposition the time is $\dfrac{300}{x+5}$ hours;

$$\therefore \quad \frac{300}{x+5}=\frac{300}{x}-2;$$

whence $$x^2+5x-750=0,$$

or $$(x+30)(x-25)=0;$$

$$\therefore \quad x=25, \text{ or } -30.$$

Hence the train travels 25 miles per hour, the negative value being inadmissible.

[For an explanation of the meaning of the negative value see Hall & Knight's *Elementary Algebra*, p. 209.]

Example IV. A cistern can be filled by two pipes in $33\frac{1}{3}$ minutes; if the larger pipe takes 15 minutes less than the smaller to fill the cistern, find in what time it will be filled by each pipe singly.

Suppose that the two pipes running singly would fill the cistern in x and $(x-15)$ minutes.

Then they will fill $\frac{1}{x}$ and $\frac{1}{x-15}$ of the cistern respectively in one minute; and, therefore, when running together, they will fill $\left(\frac{1}{x}+\frac{1}{x-15}\right)$ of the cistern in one minute.

But they fill $\frac{1}{33\frac{1}{3}}$, or $\frac{3}{100}$ of the cistern in one minute.

Hence $$\frac{1}{x}+\frac{1}{x-15}=\frac{3}{100},$$

or $$100(2x-15)=3x(x-15);$$

$$\therefore\ 3x^2-245x+1500=0,$$

or $$(x-75)(3x-20)=0;$$

$$\therefore\ x=75, \text{ or } 6\tfrac{2}{3}.$$

Thus the smaller pipe takes 75 minutes, the larger 60 minutes. The other solution $6\frac{2}{3}$ is inadmissible.

13. *A* and *B* start together to walk 10 miles; *A* walks $1\frac{1}{2}$ miles an hour faster than *B*, and arrives at his journey's end $1\frac{1}{2}$ hours sooner than *B*; at what rate did each walk?

14. A steamer goes a journey of 3240 miles: if it had gone 3 miles an hour faster it would have taken $1\frac{1}{2}$ days less: find the time the journey occupied.

15. An ordinary train, the average speed of which is 15 miles an hour less than that of the express, takes 2 hours longer to go 180 miles. What is the average speed of each train?

16. Two men start at the same time to meet each other from towns which are 54 miles apart. If one takes 3 minutes longer than the other to walk a mile, and they meet in six hours, how fast does each walk?

17. *A* sets out to meet *B* an hour after the latter starts to meet *A* from a town 38 miles away; when they meet, *B*, who has been walking at a uniform speed of half-a-mile per hour less than *A*, has gone 2 miles further than *A*: find their rates of walking.

18. A train goes a journey of 120 miles; after going half the distance at a uniform speed an accident occurs which reduces the speed by 8 miles per hour, and the train arrives $22\frac{1}{2}$ minutes late: what was its original speed?

19. A cistern can be filled by two pipes in $3\frac{3}{4}$ minutes, if the larger pipe takes 4 minutes less than the smaller to fill the cistern, find in what time it will be filled by each pipe singly.

20. Two pipes open together can fill a cistern in $9\frac{3}{8}$ minutes: if the smaller pipe takes 10 minutes more than the larger to fill the cistern, find in what time it will be filled by each pipe singly.

21. A cistern can be emptied by a pipe which takes 8 minutes more than the time required to fill the cistern by the supply pipe; if the two pipes were left open when the cistern was empty, it would be filled in 30 minutes. Suppose the cistern were full, and the supply pipe closed; in how many minutes would the cistern be emptied?

22. A fruiterer packs 420 oranges, half in boxes of a certain size, and the remainder in boxes holding 12 more each: if he requires 12 boxes in all, how many does a box of each size hold?

23. I bought a number of cricket balls for £3; if I had bought a cheaper sort costing 2*s.* apiece less, I should have had 15 more for the money: what did I pay for each?

24. A man buys a number of articles for £1, and after losing 3, gets 25*s.* by selling the rest at 9*d.* apiece more than they cost; how many did he buy?

208. We now give two examples of a somewhat more difficult character, and these should be carefully studied before the pupil attempts the miscellaneous problems in the exercise which follows.

Example I. A tradesman bought a number of yards of silk for £9. 7*s.* 6*d.*; he kept 4 yards, and sold the rest at half-a-crown per yard more than he gave, thereby obtaining £1. 2*s.* 6*d.* more than he originally spent: how many yards did he buy?

Let x be the number of yards bought.

Then the *cost* price *per yard* is $\frac{75}{x}$ half-crowns.

But the *total selling price* is £10. 10*s.*, or 84 half crowns;

$\therefore$ the *selling* price *per yard* is $\frac{84}{x-4}$ half-crowns;

$\therefore$ the *gain per yard* is $\left(\frac{84}{x-4}-\frac{75}{x}\right)$ half-crowns.

But from the question the gain per yard is 1 half-crown.

Thus $$\frac{84}{x-4}-\frac{75}{x}=1;$$

whence $$x^2-13x-300=0,$$

or $$(x-25)(x+12)=0;$$

$$\therefore\ x=25,\ \text{or}\ -12.$$

Thus the number of yards bought $=25$.

The negative value is inadmissible.

Example II. A man invests some money in 3 per cent. stock; if the price were £15 more, he would receive 1 per cent. less for his money: at what price does he buy the stock?

Let x be the number of pounds he pays for each £100 share.

Then, since £x gains £3, the percentage on his money $=\frac{3}{x} \cdot 100$.

But, if the price were £$(x+15)$, ,, ,, ,, $=\frac{3}{x+15} \cdot 100$.

Therefore the difference in the percentages $=\frac{300}{x}-\frac{300}{x+15}$.

Thus $\frac{300}{x}-\frac{300}{x+15}=1$; whence $x=60$, or -75.

Thus the stock is purchased at 60.

The negative value is inadmissible.

EXAMPLES XXV. b.

1. A tradesman bought a number of yards of silk for £6. 15*s.*; he kept 5 yards and sold the rest at 1*s.* 6*d.* per yard more than he gave, thereby gaining 15*s.* on his original outlay: how many yards did he buy?

2. A hawker buys a certain number of oranges for 5*s.*, and retails them at 8 for 6*d.*, thereby gaining as much as he paid for 25 oranges: how many did he buy?

3. For £720 a man purchased some horses; 3 of them died, and he sold the remainder at £6 apiece more than he gave, thereby gaining 5 per cent. on his outlay: how many horses did he buy?

4. A man invests some money in $3\frac{1}{2}$ per cent. stock; if the price were £15 less, he would receive $1\frac{1}{6}$ per cent. more for his money: what price does he pay for the stock?

5. A tradesman finds that by selling a book for 4*s.* 8*d.* his percentage of profit is the same as the number of pence the book cost him: what did he pay for it?

6. The sum of the ages of a father and son is 80 years, also one-twelfth of the product of their ages in years exceeds the father's age by 56. How old are they?

7. If on New Year's day each one of a family sends a card to each of the rest, and the postman delivers 156 cards, how many are there in the family?

8. The product of three consecutive integers, of which a certain number is the lowest, exceeds by 294 the product of three consecutive integers of which that number is the highest. Find the number.

9. I bought a certain number of books for 30*s.*; if each book had been subject to a discount of 4*d.*, I should have had three more for the money: find the cost of each.

10. If 6 fewer bottles of wine can be bought for £5 when the price is raised ten shillings per dozen, what is the original price?

11. A and B distribute £5 each in charity: A relieves 5 persons more than B, and B gives to each 1*s.* more than A. How many did each relieve?

12. How many pears are bought for 1*s.*, when 6 more for the money lowers the price 2*d.* a dozen?

13. The price of one kind of sugar is 1*s.* 9*d.* per stone more than that of another kind; and 8 lbs. less of the first kind than of the second can be bought for £1. Find the price of each per stone.

14. P sets out from London to meet Q, who starts at the same time from Maidstone, 35 miles distant. P walks $1\frac{1}{2}$ miles an hour faster than Q, but after two hours stops at a friend's house on the way for $1\frac{1}{2}$ hours; he then proceeds again, and meets Q half-way between London and Maidstone. Find the rate at which each walks.

15. A person selling a horse for £37. 10*s.* finds that his loss per cent. is half the number of pounds that he paid for the horse: what was the cost price?

TEST PAPERS H.

[*For Examination Purposes towards the end of the second term of the School Year.*]

I.

1. If $a=7$, $b=1$, $2c=5$, find the numerical value of

$$\frac{\sqrt{a^2+b^2+c^2}}{a-2b(a-3c+2)}+\frac{\sqrt[3]{2(3b-a)}}{3c+\frac{1}{2}}.$$

2. Substitute $\dfrac{2a^2-3b^2}{a+2b}$ for x in $\dfrac{x+2a}{x-3b}$, and reduce the result to its simplest form.

3. Find the square root of $4x^6-12x^4+28x^3+9x^2-42x+49$.

4. Solve (1) $x+y=4, \quad (a+b)x+(a-b)y=4a$;
   (2) $96x^2=4x+15$.

5. I bought a horse and carriage for £90; I sold the horse at a gain of 12 per cent. and the carriage at a loss of 4 per cent., and gained on the whole 6 per cent. What was the original cost of the carriage?

II.

1. Find the product of $\frac{1}{2}x^2 - \frac{3}{4}x + 6$ and $\frac{2}{3}x - \frac{3}{2}$.

2. Find the H.C.F. of

$$2x^4 - 3x^3 + 3x^2 - 6x - 2 \text{ and } 4x^4 - 6x^3 - 3x - 1.$$

3. Simplify $(x+3)\left\{1 + \frac{4(x+1)}{x^2 - x - 2}\right\}\left\{1 - \frac{5(x+2)}{x^2 + 5x + 6}\right\}$.

4. Solve (1) $\frac{x+3}{4} - 2 = \frac{6x-1}{24} - \frac{4x+9}{7x-11}$;

(2) $ax + by = c, \quad a^2x + b^2y = c^2$.

5. A boat travels 180 miles at a uniform rate; if the speed had been 3 miles an hour more, the journey would have taken 3 hours less: find the rate of the boat.

III.

1. Write down the value of $\left(1 - \frac{1}{2}x + \frac{2}{3}x^2\right)^2$.

2. Simplify: (1) $\dfrac{1 + \dfrac{a}{b}}{\dfrac{b}{a} - 1} \div \dfrac{a^2 - b^2}{1 - \dfrac{2b}{a} + \dfrac{b^2}{a^2}}$; (2) $\dfrac{x - 1 + \dfrac{1}{x}}{x^2 + 1 + \dfrac{1}{x^2}}$.

3. Find the square root of $a^2 - a - 6ab + 9b^2 + 3b + \frac{1}{4}$.

4. Solve the equations:

(1) $5x^2 + 4x - 1 = 0$;

(2) $y - 1 \cdot 2x = \cdot 25y + 1 \cdot 8x = 1 \cdot 4$.

5. A man buys oranges for 8*s.* 4*d.* and retails them at 15 for 6*d.*, thereby gaining as much as 60 oranges cost him. How many did he buy?

IV.

1. When $a = 2$, $b = 1$, $c = -3$, find the numerical value of

$$\frac{bc}{\sqrt{c^2 + (b-c)^2}}(a - b + c) + \frac{ac}{\sqrt{c^2 - (a-c)}}(a + b + c).$$

2. A man who owes x half-crowns tenders a £5 note in payment, and receives as change $2x$ florins, $3x$ sixpences, and 4 shillings. Find the amount of the debt.

3. Simplify $\left(1 + \frac{3n(3m+n)}{m^2 - 5mn - 24n^2}\right)\left(1 - \frac{n(m+7n)}{m^2 - 49n^2}\right)$.

4. Solve the equations:

(1) $\dfrac{x-a}{b-2a}+\dfrac{2(x-b)}{a-2b}=1$; (2) $2(x^2-3x)=3-x$.

5. A farmer bought some sheep for £72, and found that if he had received 6 more for the same money he would have paid £1 less for each. How many did he buy?

V.

1. Write down the square of $a^2-1+\dfrac{1}{a^2}$.

2. Find the product of

(1) $2a^2-3ab+5b^2$ and $2a^2-3ab-5b^2$;

(2) $1-2a^2$, $1-2a^2+4a^4$, $1+2a^2$, and $1+2a^2+4a^4$.

3. Simplify:

(1) $\dfrac{x^3-6x^2+11x-6}{2x^4-18x^3+52x^2-48x}$; (2) $\dfrac{1}{2x+1}-\dfrac{1}{3(x+\frac{1}{2})}-\dfrac{1}{6x+3}$.

4. Solve the equations:

(1) $\dfrac{3x-1}{4}-\dfrac{1}{2}(x+1)=x+1-\dfrac{1}{6}(5x+3)$;

(2) $5(x^2-4x)=8-2x$.

5. The sum of the squares of two consecutive odd numbers is 74; find them.

VI.

1. Given $3x=1$, $2y=-3$, $z=-4$, find the numerical value of the following expressions:

(1) $\dfrac{9x}{8y^2}+\dfrac{24y}{z^2}+\dfrac{z}{36x^2}$; (2) $\dfrac{3x}{y+z}+\dfrac{z}{x+y}$.

2. Resolve the following expressions into factors:

(1) $a^4+a^2b^2+b^4$; (2) $6(x-1)^2-x$; (3) $10x^3-40xy^2$.

3. Find the square root of $x^6+6x^5+13x^4+4x^3-20x^2-16x+16$.

4. Solve the following equations:

(1) $x(x-a)+x(x-b)=2(x-a)(x-b)$;

(2) $1\cdot35x+\cdot1y=12\cdot2$, $\cdot\dot{3}x-2\cdot5y=1\cdot75$.

5. Find the fraction which becomes equal to $\frac{1}{2}$ when its denominator is decreased by 6, and becomes equal to $\frac{3}{4}$ when its numerator and denominator are each increased by 124.

VII.

1. A man is 42 years of age, and the sum of the ages of his three sons is 26. How many years ago is it since the joint ages of these sons was equal to half the father's age?

2. Find the L.C.M. of x^3+2x^2-x-2, and x^3-3x+2.

3. Simplify:

$$(1)\quad \frac{x}{(x-y)(a+x)}+\frac{y}{(y-x)(a+y)};\qquad (2)\quad \frac{24x^4-20x^2-24}{8x^3+6x^2-12x-9}.$$

4. Solve the equations:

$$(1)\quad x\left(\frac{1}{a}-\frac{1}{b}\right)-\frac{2x}{a-b}+ab=0;\qquad (2)\quad \frac{5+x}{3}=\frac{7+y}{5}=\frac{9+x+y}{7}.$$

5. A man pays income-tax at 9*d.* in the £. Were his income increased 10 per cent., and the tax raised to 10*d.* in the £, his net income would be increased by £33. What is his income?

VIII.

1. Simplify

$$2x^2-3\left(\frac{y^2}{2}-x^2\right)-\frac{1}{2}\left\{4x^2-\frac{1}{2}(24x^2+5y^2)\right\},$$

and divide the simplified expression by $x+\frac{y}{6}$.

2. Shew that $x^2\left(x-2+\frac{1}{x}\right)\left(x+2+\frac{1}{x}\right)\left(1+\frac{2}{x^2}+\frac{1}{x^4}\right)=\left(x^2-\frac{1}{x^2}\right)^2$

3. Simplify:

$$(1)\quad \frac{x^2}{ab}+\frac{(x-a)^2}{a(a-b)}-\frac{(x-b)^2}{b(a-b)};\qquad (2)\quad \frac{3x^3-2x^2-x}{4x^3-2x^2-3x+1}.$$

4. Solve the equations:

$$(1)\quad \frac{2x+3}{x+1}=\frac{4x+5}{4(x+1)}+\frac{3(x+1)}{3x+1};$$

$$(2)\quad ax+by=c,\quad cx-ay=b.$$

5. What are eggs selling at when, if the price were raised threepence a dozen, one would get four fewer in a shilling's worth?

CHAPTER XXVI.

MISCELLANEOUS QUADRATIC EQUATIONS.

Quadratics with Literal Coefficients.

209. The method of Solution by Factors can be applied to great advantage in the case of quadratics which contain literal coefficients.

Example 1. Solve $2x^2 - ax + 2bx = ab$.

Transposing, *so as to have all the terms on one side of the equation,* we have

$$2x^2 - ax + 2bx - ab = 0.$$

Now $$2x^2 - ax + 2bx - ab = x(2x - a) + b(2x - a)$$
$$= (2x - a)(x + b).$$

Therefore $$(2x - a)(x + b) = 0;$$

whence $$2x - a = 0, \text{ or } x + b = 0,$$

$$\therefore\ x = \frac{a}{2}, \text{ or } -b.$$

Example 2. Solve $7(x + 2a)^2 + 3a^2 = 5a(7x + 23a)$.

Simplifying, $$7x^2 + 28ax + 28a^2 + 3a^2 = 35ax + 115a^2;$$

that is, $$7x^2 - 7ax - 84a^2 = 0;$$

$$\therefore\ x^2 - ax - 12a^2 = 0,$$

or $$(x - 4a)(x + 3a) = 0;$$

$$\therefore\ x = 4a, \text{ or } -3a.$$

EXAMPLES XXVI. a.

Solve the equations :

1. $x^2 - 3ax + 2a^2 = 0$.
2. $x^2 + ax = 6a^2$.
3. $x^2 - 4bx = 5b^2$.
4. $x^2 + 4cx = 21c^2$.
5. $x^2 = dx + 72d^2$.
6. $x^2 = 2ax + 35a^2$.
7. $2x^2 + 5ax = 3a^2$.
8. $3x^2 + 2c^2 = 7cx$.
9. $4d^2 = 6x^2 - 23dx$.
10. $(2x + a)(x - 2a) + 2x(x + 5a) = 0$.
11. $(2x - 3k)^2 = 7kx - 2(3k^2 + x^2)$.
12. $ax^2 - ab^2 + b^3 = bx^2$.
13. $2(x - a)^2 - ax = 3a(a - 2x)$.
14. $ax + 4bx = 2(x^2 + ab)$.
15. $(a - b)x^2 = (a + b)x - 2b$.

Solve the equations :

16. $x+\dfrac{a^2-b^2}{4x}-a=0.$

17. $\dfrac{x}{a}+\dfrac{a}{x}=\dfrac{b}{a}+\dfrac{a}{b}.$

18. $\dfrac{1}{2x-5c}+\dfrac{5}{2x-c}=\dfrac{2}{c}.$

19. $\dfrac{1}{6x-5a}-\dfrac{2}{a}=\dfrac{5}{a-6x}.$

20. $\dfrac{x-a}{2x-b}=\dfrac{b}{x+a}.$

21. $\dfrac{d+x}{c-2x}=\dfrac{c}{d-x}.$

22. $\dfrac{2}{3x-2m}+\dfrac{3}{2x-3m}=\dfrac{7}{2m}.$

23. $\dfrac{a^2b}{x^2}+\left(1+\dfrac{b}{x}\right)a=2b+\dfrac{a^2}{x}.$

Formation of Equations with given roots.

210. It is now easy to form an equation whose roots are known.

Example 1. Form the equation whose roots are 4 and -3.

Here $$x=4, \text{ or } \quad x=-3;$$

$$\therefore \; x-4=0, \text{ or } x+3=0;$$

both of these statements are included in

$$(x-4)(x+3)=0,$$

or $$x^2-x-12=0,$$

which is the required equation.

Example 2. Form the equation whose roots are a and $-\dfrac{b}{3}$.

Here $$x=a, \text{ or } \quad x=-\frac{b}{3};$$

$$\therefore \; x-a=0, \text{ or } x+\frac{b}{3}=0;$$

$$\therefore \text{ the equation is } (x-a)\left(x+\frac{b}{3}\right)=0;$$

that is, $$(x-a)(3x+b)=0,$$

or $$3x^2-3ax+bx-ab=0.$$

[See Hall & Knight's *Elementary Algebra*, Chapter XXXVI.]

EXAMPLES XXVI. b.

Form the equations whose roots are

1. $4, \; -3.$ 2. $-9, \; 4.$ 3. $-2, \; -7.$ 4. $11, \; 13.$

5. $\dfrac{2}{3}, \; \dfrac{3}{4}.$ 6. $\dfrac{3}{2}, \; -\dfrac{2}{3}.$ 7. $0, \; \dfrac{4}{5}.$ 8. $0, \; -\dfrac{7}{12}.$

9. $a, \; -b.$ 10. $-2a, \; 3b.$ 11. $5, \; -4a.$ 12. $a+b, \; a-b.$

13. $-a, \; \dfrac{b}{2}.$ 14. $\dfrac{3}{4}a, \; -\dfrac{4}{5}a.$ 15. $2p, \; -\dfrac{p}{2}.$ 16. $\dfrac{c}{2d}, \; \dfrac{d}{2c}.$

Solution of a Quadratic by Formula.

211. From the working of the various examples by the methods already explained it appears that, after suitable reduction and transposition, every quadratic equation can be written in the form

$$ax^2+bx+c=0,$$

where a, b, c may have *any numerical values whatever.* If therefore we can solve this quadratic we can solve any.

Transposing, $$ax^2+bx=-c;$$

dividing by a, $$x^2+\frac{b}{a}x=-\frac{c}{a}.$$

Complete the square by adding to each side $\left(\frac{b}{2a}\right)^2$; thus

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a};$$

that is, $$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2};$$

extracting the square root,

$$x+\frac{b}{2a}=\frac{\pm\surd(b^2-4ac)}{2a};$$

$$\therefore\ x=\frac{-b\pm\surd(b^2-4ac)}{2a}.$$

Thus the roots are $\frac{-b+\surd(b^2-4ac)}{2a}$ and $\frac{-b-\surd(b^2-4ac)}{2a}$.

212. We may now make use of this general formula for the roots of a quadratic; and, in adapting it to the particular case in question, we have simply to substitute the values corresponding to a, b, and c.

Example 1. Solve $3x^2+5x-2=0$.

Applying the formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, we here have $a=3$, $b=5$, $c=-2$.

$$\therefore\ x=\frac{-5\pm\sqrt{(5)^2-4\,.\,3\,.\,(-2)}}{2\,.\,3}$$

$$=\frac{-5\pm\sqrt{25-(-24)}}{6}=\frac{-5\pm\sqrt{49}}{6}=\frac{-5\pm7}{6}=\frac{1}{3}\text{ or }-2.$$

Example 2. Solve $x^2-\frac{5}{12}x-\frac{1}{6}=0.$

Here we might at once employ the formula by putting $a=1$, $b=-\frac{5}{12}$, $c=-\frac{1}{6}$; but, by first clearing of fractions, we simplify the working.

Thus we have $12x^2-5x-2=0.$

Here $a=12,\ b=-5,\ c=-2.$

$$\therefore\ x=\frac{-(-5)\pm\sqrt{(-5)^2-4\,.\,12\,.\,(-2)}}{2\,.\,12}$$

$$=\frac{5\pm\sqrt{25+96}}{24}=\frac{5\pm 11}{24}=\frac{2}{3}\text{ or }-\frac{1}{4}.$$

213. In the result $x=\frac{-b\pm\sqrt{(b^2-4ac)}}{2a}$,

it must be remembered that the expression $\sqrt{(b^2-4ac)}$ is the square root of the compound quantity b^2-4ac, *taken as a whole.* We cannot simplify the solution unless we know the numerical values of a, b, c. It may sometimes happen that these values do not make b^2-4ac a perfect square. In such a case the exact numerical solution of the equation cannot be determined.

Example. Solve $x^2-3x-2=0.$

Here $a=1,\ b=-3,\ c=-2;$

$$\therefore\ x=\frac{-(-3)\pm\sqrt{(-3)^2-4\,.\,1\,.\,(-2)}}{2\,.\,1}$$

$$=\frac{3\pm\sqrt{9+8}}{2}=\frac{3\pm\sqrt{17}}{2}.$$

Since 17 has not an exact square root, this result cannot be simplified; thus the two roots are

$$\frac{3+\sqrt{17}}{2},\ \frac{3-\sqrt{17}}{2}.$$

Note. By finding the square root of 17 to sufficient decimal places we can find any required *approximate* values of the above roots; but when the numerical values of the unknown quantity are not required, it is usual to leave the roots in the form given above.

EXAMPLES XXVI. c.

By using the formula given in Art. 211, solve the following equations :

1. $x^2-10x+24=0$.
2. $x^2-56=x$.
3. $x^2-45=4x$.
4. $2x^2=5x-3$.
5. $6x^2+10=17x$.
6. $5x+3=8x^2$.
7. $12x^2-28x+15=0$.
8. $x^2-ax=20a^2$.
9. $15x^2+2ax=8a^2$.
10. $x^2-5x-2=0$.
11. $x^2-5x+4=0$.
12. $x^2+3x+1=0$.
13. $x^2-3x=3$.
14. $4x^2=1+x$.
15. $x=4(x^2-2)$.
16. $x^2+\frac{x}{6}=\frac{1}{18}$.
17. $x^2-1=\frac{5x}{6}$.
18. $x^2+\frac{3x}{28}-\frac{9}{14}=0$.
19. $3x^2+bx=ab+3ax$.
20. $4x^2-3dx=6cd-8cx$.

TEST PAPERS K.

[*For Examination purposes towards the end of the School Year, when the work has been revised.*]

I.

1. What must be subtracted from

$$a^6-12b^6+3a^3b^3-a^5b-11a^2b^4+5ab^5$$

to make it exactly divisible by $a^2-ab+3b^2$?

2. Find the square root of $\frac{x^4}{4}-\frac{x^3}{3}+\frac{55x^2}{9}-4x+36$.

3. Simplify : (1) $\frac{2x^2-13x+18}{2x^3-11x^2+3x+27}$;

(2) $\frac{2}{1-a}+\frac{1}{1+a}+\frac{1}{a^2-1}+\frac{3}{a^2+1}-\frac{5}{1-a^4}$.

4. Solve the equations :

(1) $ax+by=ay-bx=a^2+b^2$; (2) $9x^2-18x=16$;

(3) $\frac{2(x^2-1)}{5}+\frac{15}{2}\left(1-\frac{x^2}{3}\right)+\frac{19}{10}=\frac{9}{5}\left(\frac{x^2}{6}-\frac{1}{3}\right)$.

5. A man bought a number of articles which he sold, half at 5 per cent. profit, and the rest at 4*s.* 6*d.* each. If he loses $2\frac{1}{2}$ per cent. on the whole, what was the cost of each article?

6. Divide 16 into two parts such that their product added to the sum of their squares may be 208.

II.

1. Divide $\frac{1}{2}\left[n^2\left\{n^2-\frac{np}{6}\left(1+\frac{p^3}{n^3}\right)-\frac{31}{36}p^2\right\}\right]+\frac{11}{36}np^3$

by $$\frac{n}{3}(2n+p)-\frac{p^2}{2}.$$

2. Find the H.C.F. of $5x+4+3x^3+4x^2$ and $2x^3+x-2+5x^2$.

3. Simplify:

(1) $\left(\frac{1}{x^3-y^3}-\frac{1}{x^3+y^3}\right)\div\left(\frac{1}{x-y}-\frac{1}{x+y}-\frac{2y}{x^2+y^2}\right)$;

(2) $\left\{\frac{2x^2+x-6}{4x^2-x-5}\div\frac{(2x-3)(x^2-4)}{4x^2+7x-15}\right\}-\frac{x^2+6x+9}{(x+1)(x-2)(x+3)}$.

4. Solve the equations:

(1) $(a-c)(x+b)+(a-b)(x+c)=\frac{1}{2}(b-c)^2$;

(2) $\frac{7x+3}{5}=\frac{5y-7}{4}=\frac{2x+3y-5}{3}$; (3) $\frac{x+1}{x+2}+\frac{x+2}{x+3}=1\frac{5}{12}$.

5. A certain number of persons paid a bill; if there had been ten more, each would have paid 2*s.* less; if there had been five less, each would have paid 2*s.* 6*d.* more: find the number of persons and what each had to pay.

6. *A* runs at the rate of one mile per hour faster than *B*. He gives *B* a start of 88 yards in a mile race and beats him by 20 seconds. Find the rate at which each runs.

III.

1. Shew that $x+y$ and $x-y$ are factors of the expression
$$(x^2-2y^2)^3+(2x^2-y^2)^3.$$

2. Find the square root of $\frac{x^4}{4}+\frac{b^4}{16}-2ab+\frac{4ax^2}{b}-\frac{b^2x^2}{4}+\frac{16a^2}{b^2}$.

3. Find the value of $x-4$

(1) $\frac{(x^2+4x+3)(x^2-6x+8)}{(x^2+x-6)(x^2+x)}$; (2) $\frac{3a^3-3a^2b+ab^2-b^3}{4a^2b-5ab^2+b^3}$;

(3) $\left(1+\frac{x^2-y^2-z^2}{2yz}\right)\times\left(\frac{1}{x+y-z}+\frac{1}{z+x-y}\right)$.

4. Solve the equations:

(1) $\frac{1}{2}-\left\{\frac{1}{3}-\frac{1}{4}(1-\overline{x-2})\right\}=\frac{1}{3}-\left\{\frac{1}{4}-\frac{1}{2}(1-\overline{x-1})\right\}$;

(2) $150x^2=299x+2$; (3) $2x(3x-2a)=a(x-a)$.

5. There are two numbers of which the sum is 70. If the first be divided by the second the quotient is 2 and the remainder 1. Find the numbers.

6. A man bought some pigs for £4. 16*s.*, and sold them again at 13*s.* 6*d.* per head, gaining by the transaction as much as one pig cost him. What number did he buy?

IV.

1. Write down the values of

(1) $\left(1+\frac{1}{2}x+\frac{1}{3}x^2\right)^2$; (2) $\left(\frac{1}{2}a^2-\frac{2}{3}b^3\right)^3$.

2. If $x+3$ divides $3x^2-x-6a$ without remainder, find the value of a.

3. Simplify:

(1) $1-\dfrac{1}{1-\dfrac{1}{1-a}}\times\dfrac{1-\dfrac{1}{a}}{1-\dfrac{1}{1-\dfrac{1}{1-a}}}$; (2) $\dfrac{m-n}{m-\dfrac{2n^2}{m+n}}\div\dfrac{m+n}{\dfrac{m}{n}+2}$.

4. Solve the equations:

(1) $\dfrac{75-x}{3(x+1)}+\dfrac{80x+21}{5(3x+2)}=\dfrac{23}{x+1}+5$;

(2) $\frac{1}{2}(y-3x)+\frac{1}{5}(x+y)=3,\qquad \frac{1}{5}(3y-2x)-\frac{1}{3}(x-y)=6$;

(3) $a^2x-b^3=(x-b)(ax+bx+b^2)$.

5. Divide £64 among three persons so that the first may have three times as much as the second, and the third one-third as much as the first and second together.

6. *A* and *B* set out together along a road at the rate of $3\frac{1}{2}$ and $3\frac{3}{4}$ miles per hour respectively; 3 hours later *C* starts from the same place, and overtakes *B* two hours after passing *A*. Find *C*'s rate of travelling.

V.

1. Find the numerical value of $\dfrac{a-b}{a+b}+\sqrt{2-\dfrac{a}{b}-\dfrac{b}{a}}$, when $8a=1$, and $2b=-1$.

2. Find the square root of $\dfrac{x^2}{y^2}+\dfrac{4z}{5y}-\dfrac{2x}{3z}+\dfrac{4z^2}{25x^2}-\dfrac{4y}{15z}+\dfrac{y^2}{9z^2}$.

3. Simplify:

(1) $\dfrac{(ab-a-b+1)(ab+a+b+1)}{(ab-2a+b-2)(ab-a-2b+2)}$; (2) $\dfrac{x^4-x^2-2}{x^5-2x^4-2x^3+x^2+6}$;

(3) $\left(x-y-\dfrac{4y^2}{x-y}\right)\left(x+y-\dfrac{4x^2}{x+y}\right)\div\left\{3(x+y)-\dfrac{8xy}{x-y}\right\}$.

4. Solve the equations :

(1) $\dfrac{x}{6}-\dfrac{x-\frac{1}{2}}{3}-\dfrac{1}{3}\left(\dfrac{2}{5}-\dfrac{x}{3}\right)=0$; (2) $x=6(1-2x^2)$;

(3) $2x-3y+z+1=5x-3z-6=3x+2y-4z=0$.

5. If two numbers are each divided by their G.C.M. the quotients are 2 and 5 ; if both the numbers are increased by 13, and then divided by their G.C.M., the quotients are 9 and 16. Find the numbers.

6. A dealer sells a quantity of eggs. If he had sold 125 dozen more at one penny a dozen less, or if he had sold 100 dozen less at a penny a dozen more, he would have received the same amount for his eggs. Find the number of dozens and the price per dozen.

VI.

1. Shew that $x(x+1)(x+2)(x+3)+1=(x^2+3x+1)^2$.

2. Resolve into factors :

(1) $x^2+1+\dfrac{1}{x^2}$; (2) $\dfrac{3}{4}x^2y^2-\dfrac{1}{3}$; (3) $4a^2-65a+126$.

3. Simplify :

(1) $\dfrac{\dfrac{x}{y}+\dfrac{y}{x}-1}{\dfrac{x^2}{y^2}+\dfrac{x}{y}+1}\times\dfrac{1+\dfrac{y}{x}}{x-y}\div\dfrac{1+\dfrac{y^3}{x^3}}{\dfrac{x^2}{y}-\dfrac{y^2}{x}}$; (2) $\dfrac{a^2}{c+\dfrac{a^2}{c+\dfrac{a^2}{c}}}$.

4. Solve the equations :

(1) $\dfrac{x^2+10x+26}{x+5}-\dfrac{9(x-1)}{3x-4}=\dfrac{x^2+10x+22}{x+3}-\dfrac{15x-47}{3x-10}$;

(2) $(a-b)x=(a+b)y,\ x+y=c$; (3) $12x^2-12x=45$.

5. A man walks half the distance from A to B at the rate of a miles an hour, and the other half at b miles an hour. If he had ridden at the uniform rate of c miles an hour he could have covered the distance from A to B and back again in the same time. Prove that $4ab=c(a+b)$.

6. A number of persons pay a bill of £19. 13*s.* 9*d.* If there were four persons fewer they would have to pay 3*s.* a head more. How many persons are there?

VII.

1. Simplify $a^2(b-c)+b^2(c-a)+c^2(a-b)+(b-c)(c-a)(a-b)$.

2. Find the H.C.F. of

$$2x^4-3x^3+4x^2-9x-6 \text{ and } 4x^3-9x^2+12x-27,$$

and write down these expressions in factors.

3. Find the value of $\dfrac{1}{a(x+2by+z)}$ when $(a-b)x=1$, $(a^2-b^2)y=1$, and $(a+b)z=1$.

4. Solve the equations:

$$(1)\quad \frac{6x-7}{9}+x-3\tfrac{1}{2}=\frac{2x-3}{7}+\frac{1}{3}\left(4x-11\tfrac{1}{21}\right);$$

$$(2)\quad x-4y=7,\qquad \frac{x}{3y}+\frac{11}{10}=\frac{4x-5y}{5y};$$

$$(3)\quad \frac{2x+5}{3x-2}+\frac{3}{4}=\frac{2x+7}{3x-4}.$$

5. Of the candidates in a certain examination 64 per cent. passed. If there had been 11 more candidates, 9 of whom failed, the successes would have been 62·5 per cent.: how many candidates were there?

6. There is a number whose three digits, from left to right, are in descending order of magnitude and differ from each other in succession by the same amount. If the number be divided by the sum of its digits, the quotient is 70; and if from the number 396 be subtracted, the digits of the difference are the same as in the original number, but in reverse order; find the number.

VIII.

1. Find the value of $(a-b-c)^2-4bc$; observe the form of your result, and then resolve $x^4+y^4+z^4-2x^2y^2-2y^2z^2-2z^2x^2$ into factors.

2. Find the square root of $(2x^2-5x-3)(4x^2+12x+5)(2x^2-x-15)$.

3. Simplify

$$\frac{ax-a^2}{bx+ab}\times\frac{x^2+(a+b)x+ab}{x^2-(a+b)x+ab}\div\frac{a+\dfrac{ax+ab}{c}+\dfrac{ab}{x}}{b-\dfrac{b^2(x+c)}{cx}+\dfrac{bx}{c}}.$$

4. Solve the equations :

$$(1)\quad \frac{2(8-x)}{x-3}+\frac{100-13x}{12-x}=11-\frac{18}{x-3};$$

$$(2)\quad \frac{1}{2}(4x+5y)=20(x-y),\qquad \frac{1}{6}(4x-2y)+2y=\frac{1}{2};$$

$$(3)\quad \frac{2x+5}{x+2}+\frac{2x-5}{x-2}-\frac{4x-5}{x-1}=0.$$

5. A letter carrier has a hours allowed him for going from A to B and back again, including c hours for a rest at B ; but he finds that he can get b hours for rest by going d miles an hour faster each way : find his ordinary speed and the distance from A to B.

6. The bricks in a certain rectangular stack are re-arranged so that its height is doubled, and its length and breadth diminished by 2 yards and 4 feet respectively. The length, breadth, and height of the stack are then all equal. What were its original dimensions?

IX.

1. Shew that

$$(2x-1)(2x-3)(2x-5)(2x-7)=(4x^2-16x+11)^2-16.$$

2. Find the value of

$$(a+b+c)^2+(a+b-c)^2+(a-b+c)^2+(b+c-a)^2.$$

3. Find the square root of $\frac{3x}{y}\left(2+\frac{3x}{y}\right)+\frac{y}{x}\left(1+\frac{y}{4x}\right)+4.$

4. Solve the equations :

$$(1)\quad \frac{4\cdot05}{9x}-\frac{\cdot3}{\cdot8-2x}=\frac{1\cdot8}{x}-\frac{3\cdot6}{2\cdot4-6x};$$

$$(2)\quad \frac{x}{2}+\frac{2}{y}=\frac{5}{4},\qquad \frac{x}{3}+\frac{3}{y}=\frac{5}{3};\qquad\qquad (3)\quad \frac{1}{a}+\frac{1}{b}+\frac{2}{x}=\frac{2}{2(a+b)+x}.$$

5. Two passengers have together 5 cwt. of luggage, and are charged for the excess above the weight allowed 5*s.* 2*d.* and 9*s.* 10*d.* respectively; if the luggage had all belonged to one of them, he would have had to pay 19*s.* 2*d.* Find how much luggage each passenger is allowed without charge.

6. The area of a rectangular piece of carpet is 168 square feet; that of another, which is 3 feet longer, and 2 feet broader, is 238 square feet : find the length and breadth of each carpet.

EASY GRAPHS.

Application to Simultaneous Equations.

214. When there is only one simple equation connecting x and y, it is possible to find as many pairs of values of x and y as we please which satisfy the given equation. We now see that this is equivalent to saying that we may find as many points as we please on any given straight line. If, however, we have two *simultaneous* equations between x and y, there can only be one pair of values which will satisfy both equations. This is equivalent to saying that two straight lines can have only one common point.

Example. *Solve graphically the equations:*

(i) $3y-x=6$, (ii) $3x+5y=38$.

In (i) the intercepts on the axes are -6, 2. Thus the line is found by joining P $(-6, 0)$ and P′ $(0, 2)$.

In (ii) when $x=1$, $y=7$, and when $y=1$, $x=11$.

Thus the line is found by joining Q (1, 7) and Q′(11, 1).

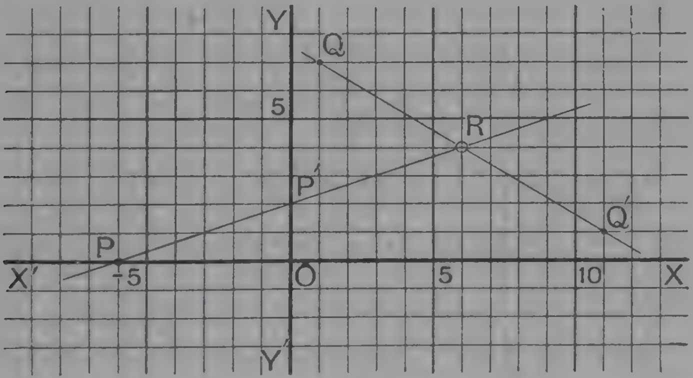

Fig. 11.

It is seen from the diagram that these lines intersect at the point R whose co-ordinates are 6, 4. Thus the solution of the given equations is $x=6$, $y=4$.

The pupil should verify this result by solving the equations algebraically by any of the methods applicable to simultaneous equations.

215. It will now be seen that the process of solving two linear simultaneous equations is equivalent to finding the coordinates of the point at which their graphs meet.

EXAMPLE. *Draw the graphs of*

(i) $5x+6y=60$, (ii) $6y-x=24$, (iii) $2x-y=7$;

and shew that they represent three lines which meet in a point.

In (i) when $y=0$, $x=12$; when $x=0$, $y=10$.

Thus the intercepts on the axes are 12 and 10, and the graph is the line PP′.

In (ii) when $x=0$, $y=4$; when $x=12$, $y=6$, and the graph is the line joining Q (0, 4) to Q′ (12, 6).

In (iii) when $x=0$, $y=-7$; when $x=8$, $y=9$, and the graph is the line joining R (0, −7) to R′ (8, 9).

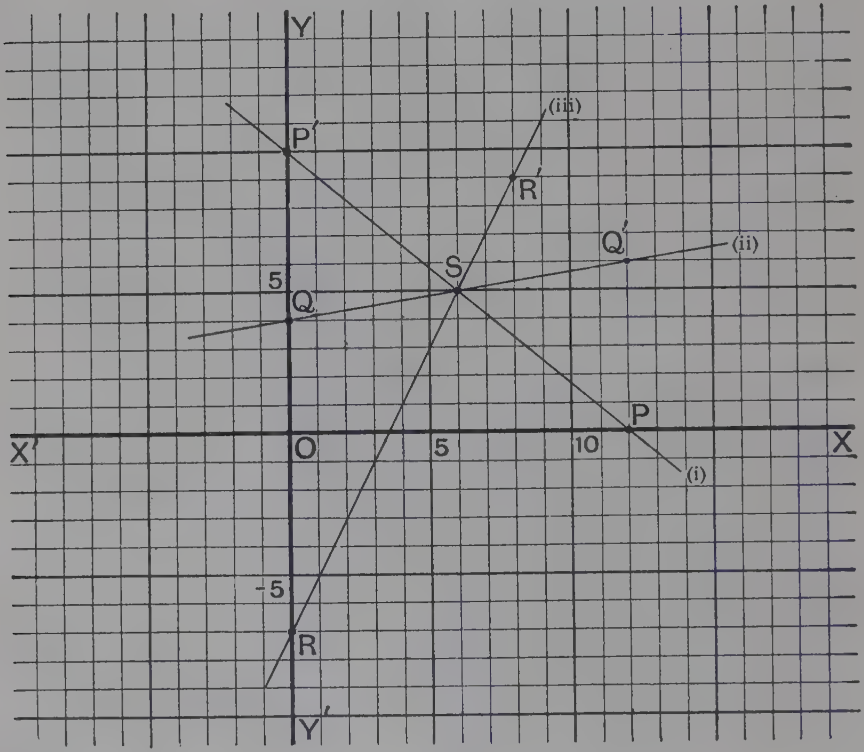

FIG. 12.

From the diagram it is evident that these three lines all pass through the point S whose coordinates are 6, 5.

216. Two simultaneous equations lead to no finite solution if they are inconsistent with each other. For example, the equations

$$x+3y=2, \quad 3x+9y=8$$

are inconsistent, for the second equation can be written $x+3y=2\frac{2}{3}$, which is clearly inconsistent with $x+3y=2$. The graphs of these two equations will be found to be two parallel straight lines which have no finite point of intersection.

Again, two simultaneous equations must be independent. The equations

$$4x+3y=1, \quad 16x+12y=4$$

are not independent, for the second can be deduced from the first by multiplying throughout by 4. Thus *any pair of values* which will satisfy one equation will satisfy the other. Graphically these two equations represent two coincident straight lines which of course have an unlimited number of common points.

EXAMPLES G. 7.

Solve the following equations graphically :

1. $y=2x+3$, $y+x=6$.

2. $y=3x+4$, $y=x+8$.

3. $y=4x$, $2x+y=18$.

4. $2x-y=8$, $4x+3y=6$.

5. $3x+2y=16$, $5x-3y=14$.

6. $6y-5x=18$, $4x=3y$.

7. $2x+y=0$, $y=\frac{4}{3}(x+5)$.

8. $2x-y=3$, $3x-5y=15$.

9. $2y=5x+15$, $3y-4x=12$.

10. Shew that the straight lines given by the equations

$$9y=5x+65, \quad 5x+2y+10=0, \quad x+3y=11,$$

meet in a point. Find its coordinates.

11. Prove by graphical representation that the three points (3, 0), (2, 7), (4, −7) lie on a straight line. Where does this line cut the axis of y?

12. Draw the graphs of the equations :

$$3\cdot4x+5y=17, \quad x-y=0\cdot8, \quad y-0\cdot5x=0\cdot45 ;$$

and shew that they all pass through one point.

13. Draw the triangle whose sides are represented by the equations :

$$3y-x=9, \quad x+7y=11, \quad 3x+y=13 ;$$

and find the coordinates of the vertices.

217. We shall now give some graphs of functions of higher degree than the first.

EXAMPLE. *Draw the graph of* $y=x^2$.

This is one of the most useful and interesting graphs the pupil will meet with; it is, therefore, important to plot the curve carefully on a suitable scale.

Take 0·4 of an inch as unit for x, and 0·1 of an inch for y, then *positive* values of x and y may be tabulated as follows:

x	0	0·5	1	1·5	2	2·5	3	3·5	4	...
y	0	0·25	1	2·25	4	6·25	9	12·25	16	...

Now if we take the following *negative* values of x

$$-0·5,\ -1,\ -1·5,\ -2,\ -2·5,\ -3,\ -3·5,\ -4,\ \ldots$$

we shall obtain the same series of values for y as before.

If the points we have now determined are plotted and connected by a continuous line drawn freehand, we shall obtain the curve shewn in Fig. 13. This curve is called a **parabola.**

There are three facts to be specially noted in this example.

(i) Since from the equation we have $x=\pm\sqrt{y}$, it follows that for every value of the ordinate we have two values of the abscissa, *equal in magnitude and opposite in sign.* Hence the graph is symmetrical with respect to the axis of y; so that after plotting with care enough points to determine the form of the graph in the first quadrant, its form in the second quadrant can be inferred without actually plotting any points in this quadrant. At the same time, in this and similar cases beginners are recommended to plot a few points in each quadrant through which the graph passes.

(ii) We observe that all the plotted points lie above the axis of x. This is evident from the equation; for since x^2 must be positive for all values of x, every ordinate obtained from the equation $y=x^2$ must be positive.

In like manner the pupil may shew that the graph of $y=-x^2$ is a curve similar in every respect to that in Fig. 13, but lying entirely below the axis of x.

(iii) As the numerical value of x increases that of y increases very rapidly. Hence, as there is no limit to the values which may be selected for x, it follows that the curve extends upwards and outwards to an infinite distance in both the first and second quadrants.

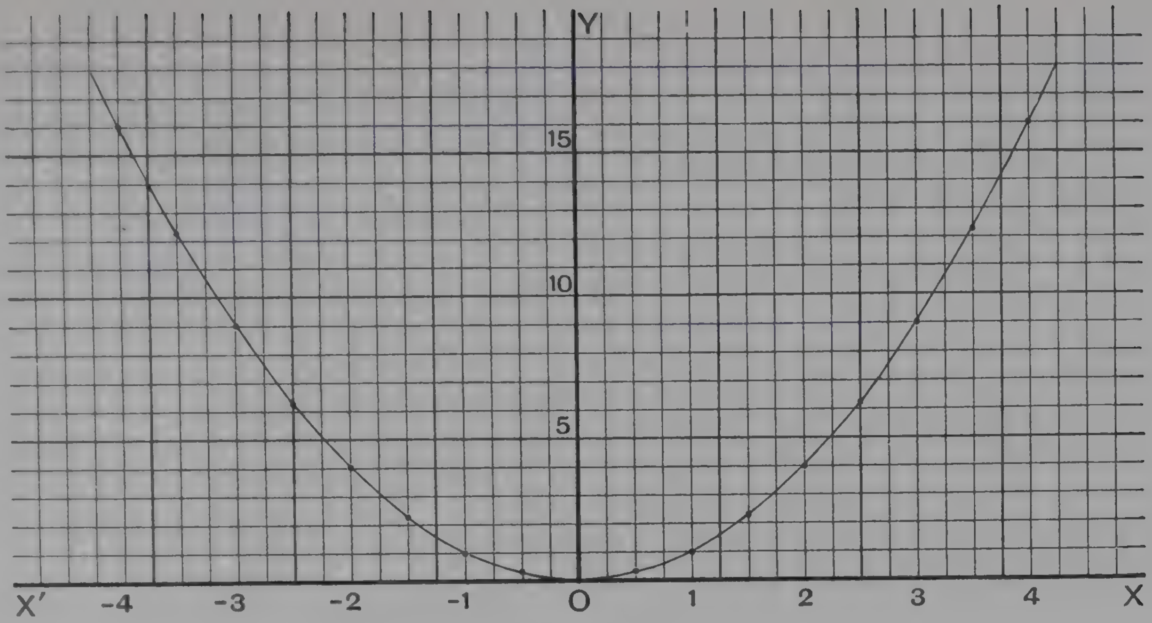

FIG. 13.

The pupil should draw this graph for himself with other units; for example, half an inch may be taken as unit on each axis. Or 1 inch may be taken as unit for x, and 0·1 of an inch as unit for y.

EXAMPLE. *Draw the graph of* $y=3-4x-4x^2$. *Thence find the roots of the equation* $4x^2+4x-3=0$. *Shew that the expression* $3-4x-4x^2$ *is positive for all real values of* x *between* $0{\cdot}5$ *and* $-1{\cdot}5$, *and negative for all real values of* x *outside these limits. Also find the greatest numerical value the expression* $3-4x-4x^2$ *can have.*

Take the unit for x four times as great as that for y, and use the following table of values:

x	2	1·5	1	0·5	0	–0·5	–1	–1·5	–2	–2·5
$-4x$	–8	–6	–4	–2	0	2	4	6	8	10
$-4x^2$	–16	–9	–4	–1	0	–1	–4	–9	–16	–25
y	–21	–12	–5	0	3	4	3	0	–5	–12

After plotting these points we have the graph given in Fig. 14.

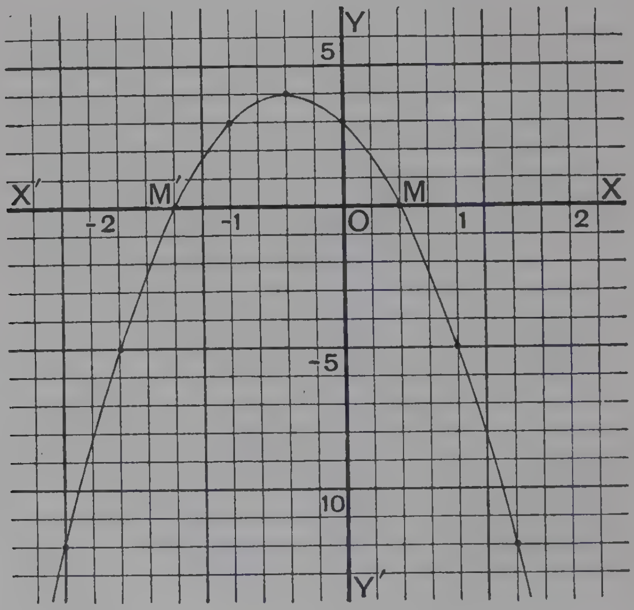

FIG. 14.

To solve the equation $4x^2+4x-3=0$ graphically we proceed as follows: the required roots are the values of x which make $4x^2+4x-3$ equal to 0. But since $y=3-4x-4x^2$, the values required are those which make $y=0$. These are found at the points M and M′ where the curve cuts the x-axis. Thus the required roots are $0{\cdot}5$ and $-1{\cdot}5$.

Again between the points M and M′ the graph lies above the x-axis; that is, the value of y, or $3-4x-4x^2$, is positive so long as x lies between 0·5 and -1·5, and is negative for other values of x.

The maximum value of the expression $3-4x-4x^2$ is the value of the greatest ordinate in the graph, namely 4.

NOTE. In the graph of $y=x^2$ (Fig. 13) it will be noticed that as we pass from right to left along the curve the ordinate is constantly decreasing until it becomes zero at O; after this the ordinate begins to increase. The point at which this change takes place in a graph is known as a **turning point**. Thus the origin is a turning point of $y=x^2$. Again in Fig. 14 there is a turning point at the point $(-0·5, 4)$.

There is always a maximum or minimum value of the ordinate at a turning point.

EXAMPLES G. 8.

1. Draw the graphs of (i) $y=x^2$, (ii) $y=8x^2$.

In (i) take 0·4″ as unit for x, 0·2″ as unit for y.

In (ii) …… 1″ ………… x, 0·1″ ………… y.

2. On the same scale as in Ex. 1. (ii) draw the graph of $y=16x^2$.

3. Plot the graph of $y=x^2$, taking 1 inch as unit on both axes, and using the following values of x.

$-0·4$, $-0·3$, $-0·2$, $-0·1$, 0, 0·1, 0·2, 0·3, 0·4.

4. Draw the graphs of

(i) $y^2=-4x$; (ii) $y=2x-\frac{x^2}{4}$; (iii) $y=\frac{x^2}{4}+x-2$.

In each case give the coordinates of the turning points.

5. Shew (i) graphically, (ii) algebraically, that the line $y=2x-3$ meets the curve $y=\frac{x^2}{4}+x-2$ in one point only. Find its coordinates.

6. Find graphically the roots of the following equations to 2 places of decimals:

(i) $\frac{x^2}{4}+x-2=0$; (ii) $x^2-2x=4$; (iii) $4x^2-16x+9=0$.

7. Find the minimum value of x^2-2x-4, and the maximum value of $5+4x-2x^2$.

8. Shew graphically that the expression x^2-2x-8 is negative for all values of x between -2 and 4, and positive for all values of x outside these limits.

Miscellaneous Applications of Graphs.

218. When two quantities x and y are so related that a change in one produces a proportional change in the other, their variations can always be expressed by an equation of the form $y=ax$, where a is some constant quantity. Hence in all such cases the graph which exhibits their variations is *a straight line through the origin*, so that in order to draw the graph it is only necessary to know the position of one other point on it. Such examples as deal with work and time, distance and time (when the speed is uniform), quantity and cost of material, principal and simple interest at a given rate per cent., may all be illustrated by linear graphs through the origin.

EXAMPLE 1. *At* 8 a.m. *A starts from* P *to ride to* Q *which is* 48 *miles distant. At the same time B sets out from* Q *to meet A. If A rides at* 8 *miles an hour, and rests half an hour at the end of every hour, while B walks uniformly at* 4 *miles an hour, find graphically*

(i) *the time and place of meeting;*

(ii) *the distance between A and B at* 11 *a.m.;*

(iii) *at what time they are* 14 *miles apart.*

In Fig. 15, on the opposite page, let the position of P be chosen as origin; let time be measured horizontally from 8 a.m. (1 inch to 1 hour), and let distance be measured vertically (1 inch to 20 miles).

In 1 hr. A rides 8 mi.; therefore the point D (1, 8) marks his position at 9 a.m. In the next half-hour he makes no advance towards Q; therefore the corresponding portion of the graph is DE. The details of A's motion may now be completed by the broken line PDEFGHKX.

On the vertical axis mark PQ to represent 48 mi. and mark the hours on the horizontal line through Q. At 9 a.m. B has walked 4 mi. towards P. Measuring a distance to represent 4 mi. *downwards* we get the point R, and QR produced is the graph of B's motion. It cuts A's graph at X. Hence the point of meeting is X, which is 28 mi. from P, and the time is 1 p.m.

The distance between A and B at any time is shewn by the difference of the ordinates. Thus at 11 a.m. their distance apart is MG, which represents 20 mi.

Lastly, NT represents 14 mi.; thus A and B are 14 mi. apart at 11.30 a.m.

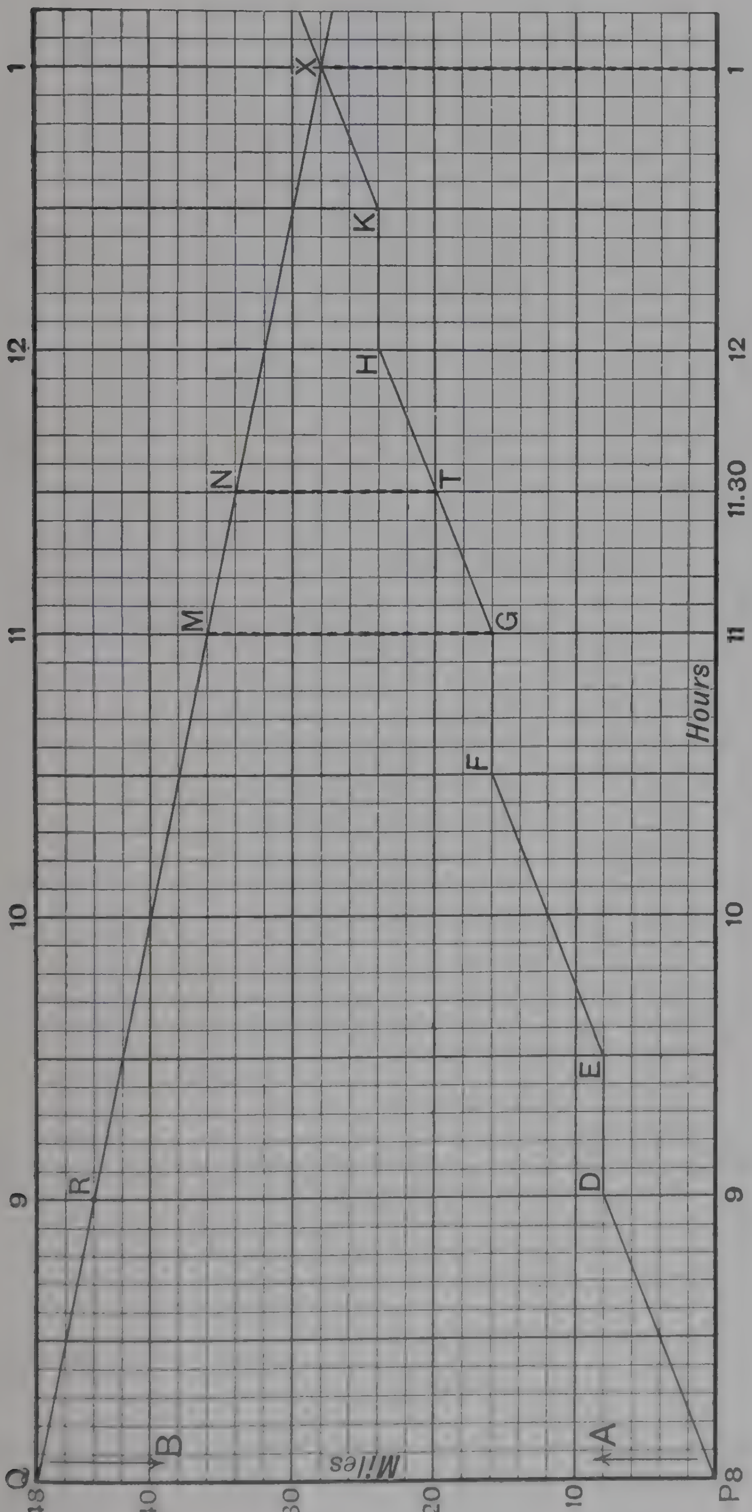

FIG. 15.

219. In examples of this type the advantage of the graphical method lies in the fact that a graph records a number of allied results which can be interpreted at a glance instead of making each a subject of separate calculation. For example, the diagram on the preceding page gives the relative positions of **A** and **B** at any time between 8 a.m. and 1 p.m.

Example 2. *A can beat B by* 20 *yards in* 120, *and B can beat C by* 10 *yards in* 50. *Supposing their rates of running to be uniform, find graphically how much start A can give C in* 120 *yards so as to run a dead heat with him. If A, B, and C start together, where are A and C when B has run* 80 *yards?*

In Fig. 16 distance is measured on the horizontal axis (1 inch to 40 yards). **OY** is taken on the vertical axis of any convenient length and represents the time taken by *A* to run 120 yards.

In the same time B and *C* would run 100 yards and 80 yards respectively. Hence the graphs of the three runners are found by joining **O** to the points marked **A**, **B**, **C** in the figure, and it is evident that *A* can give *C* a start of 40 yards in 120.

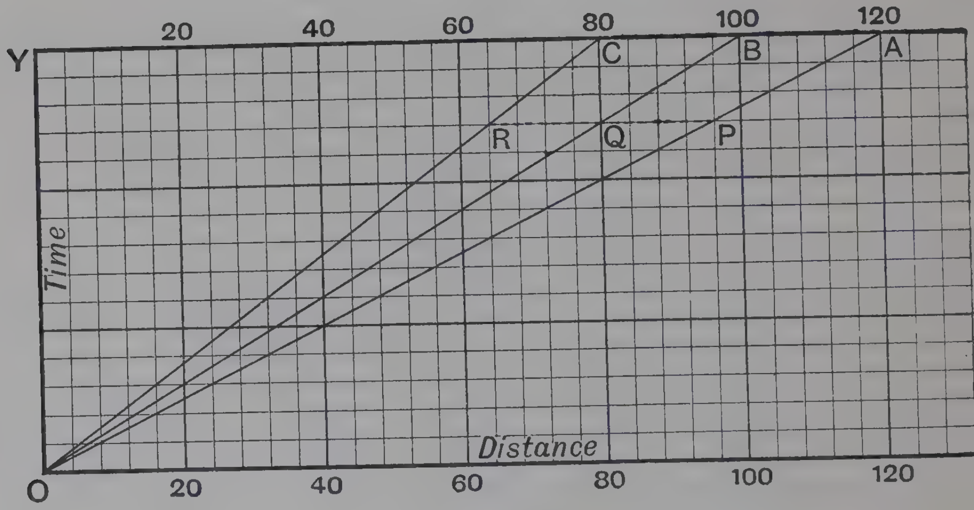

Fig. 16.

Find the point **Q** on **OB** which corresponds to an abscissa 80 yards, and draw a horizontal line cutting **OC**, **OA** in **R** and **P**. Then the distances run by *C* and *A* when *B* has gone 80 yards are given by the abscissae of **R** and **P**.

Thus *A* was 16 yards ahead, *C* 16 yards behind.

EXAMPLES G. 9.

1. By measuring time along OX (1 inch for 1 hour), and distance along OY (1 inch for 10 miles) shew that a line may be drawn from O through the points (1, 8), (2, 16), (3, 24), ... to indicate distance travelled towards Y in a specified time at 8 miles an hour.

A starts from London at noon at 8 miles an hour; two hours later *B* starts, riding at 12 miles an hour. Find graphically at what time and at what distance from London *B* overtakes *A*. At what times will *A* and *B* be 8 miles apart? If *C*, starting at 3 p.m., rides after *B* at 15 miles an hour, find from the graphs

(i) the distances between *A*, *B*, and *C* at 5 p.m.;
(ii) the time when *C* is 8 miles behind *B*.

2. With the same conditions as in Ex. 1, shew how to draw a line from Y to indicate distance travelled from Y towards O at 6 miles an hour.

If O and Y represent two towns 45 miles apart, and if *A* walks from Y to O at 6 miles an hour while *B* walks from O to Y at 4 miles an hour, both starting at noon, find graphically their time and place of meeting.

Also read off from the graphs

(i) the times when they are 15 miles apart;
(ii) *B*'s distance from Y at 6.15 p.m.

3. At noon *A* starts to walk at 6 miles an hour, and at 1.30 p.m. *B* follows on horseback at 8 miles an hour. When will *B* overtake *A*? Also find

(i) when *A* is 5 miles ahead of *B*;
(ii) when *A* is 3 miles behind *B*.

4. If two men starting at noon, 60 miles apart, ride towards each other at 12 miles and 9 miles an hour respectively, find graphically when they are first 18 miles from each other. Also find (to the nearest minute) their time of meeting.

5. Two bicyclists ride to meet each other from two places 95 miles apart. *A* starts at 8 a.m. at 10 miles an hour, and *B* starts at 9.30 a.m. at 15 miles an hour. Find graphically when and where they meet, and at what times they are $37\frac{1}{2}$ miles apart.

6. *A* and *B* start at the same time from London to Blisworth, *A* walking 4 miles an hour, *B* riding 9 miles an hour. *B* reaches Blisworth in 4 hours, and immediately rides back to London. After 2 hours' rest he starts again for Blisworth at the same rate. How far from London will he overtake *A*, who has in the meantime rested $6\frac{1}{4}$ hours?

R.B.A.N.M's

7. At what distance from London, and at what time, wi train which leaves London for Rugby at 2.33 p.m., and goes at t rate of 35 miles an hour, meet a train which leaves Rugby 1.45 p.m. and goes at the rate of 25 miles an hour, the distanc between London and Rugby being 80 miles?

Also find at what times the trains are 24 miles apart, and how far apart they are at 4.9 p.m.

8. *A*, *B*, and *C* run a race of 200 yards. *A* gives *B* a start of 8 yards, and *C* starts some seconds after *A*. *A* runs the distance in 25 seconds and beats *C* by 40 yards. *B* beats *A* by 1 second, and when he has been running 15 seconds, he is 48 yards ahead of *C*. Find graphically how many seconds *C* starts after *A*. Shew also from the graphs that if the three runners started level they would run a dead heat. [Take 1 inch to 40 yards, and 1 inch to 10 seconds.]

9. A cyclist has to ride 75 miles. He rides for a time at 9 miles an hour and then alters his speed to 15 miles an hour covering the distance in 7 hours. At what time did he change his speed?

10. *A* and *B* ride to meet each other from two towns X and Y which are 60 miles apart. *A* starts at 1 p.m., and *B* starts 36 minutes later. If they meet at 4 p.m., and *A* gets to Y at 6 p.m., find the time when *B* gets to X. Also find the times when they are 22 miles apart. When *A* is half-way between X and Y, where is *B*?

11. *A*, *B*, and *C* run a race of 300 yards. *A* and *C* start from scratch, and *A* covers the distance in 40 seconds, beating *C* by 60 yards. *B*, with 12 yards' start, beats *A* by 4 seconds. Supposing the rates of running in each case to be uniform, find graphically the relative positions of the runners when *B* passes the winning post. Find also by how many yards *B* is ahead of *A* when the latter has run three-fourths of the course.

12. A party of tourists set out for a station 3 miles distant and go at the rate of 3 miles an hour. After going half a mile one of them has to return to the starting point; at what rate must he now walk in order to reach the station at the same time as the others?

13. A motor car on its way to Bristol overtakes a cyclist at 9 a.m.; the car reaches Bristol at 10.30 and after waiting 1 hour returns, meeting the cyclist at noon. Supposing the speeds of car and cyclist to be uniform, find when the cyclist will reach Bristol. Also compare the speeds of the car and cyclist.

14. Two trains start at the same time, one from Liverpool to Manchester, and the other in the opposite direction, and running steadily complete the journey in 42 min. and 56 min. respectively. How long is it from the moment of starting before they meet?

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